Graph Theory II - Planar Graphs

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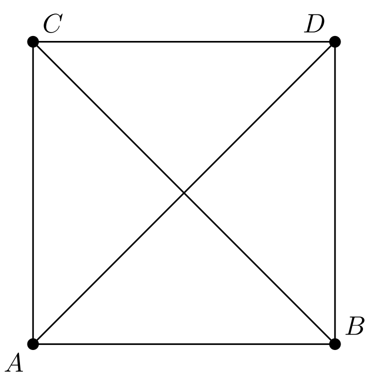
1 Spanning Trees

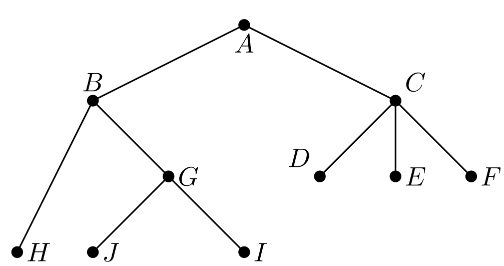
Recall from last week that a tree is a connected simple graph with no cycles.

Definition 1 *Given a graph G, a **subgraph** of G is a graph H such that $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. H is said to **span** G if H contains all the vertices of G. H is a **spanning tree** of G if it spans G and is also a tree.*

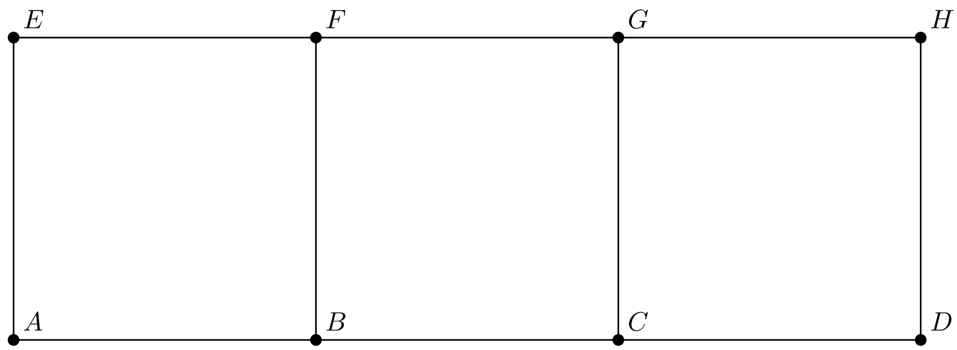
Problem 1 *For each graph below, draw one spanning tree (there may be more than one!)*



Solution: There are many spanning trees.



Solution: The graph is a tree, so there is only one possible spanning tree.



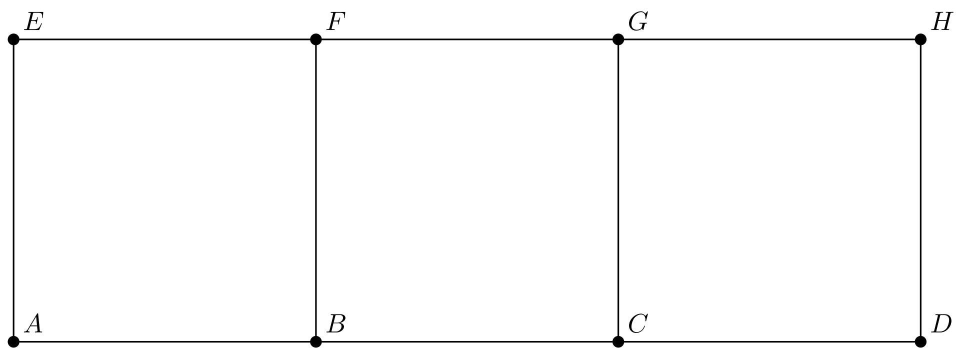
Solution: There are many spanning trees.

As one may begin to suspect, it is in fact the case that all graphs have spanning trees. To prove this, it suffices to provide an algorithm which gives a spanning tree.

Definition 2 *The **depth-first search** algorithm for a connected simple graph G is given by*

1. *Choose a vertex of G, called the node, and add it to a subgraph H.*
2. *Starting at a vertex, travel along an edge to any vertex not previously visited. Add both the edge and the vertex to H.*
3. *Repeat Step 2 until a vertex is reached whose neighbors have all been previously visited. Return to the node when this occurs.*
4. *Repeat Steps 2 and 3 until all neighbors of the node have been visited.*

Problem 2 *Use depth-first search (with the node at A) to find a spanning tree of the following graph.*



Solution: There are many spanning trees, because there are still many choices which can be made in Step 2.

Problem 3 *Explain why depth-first search always results in a spanning tree H.*

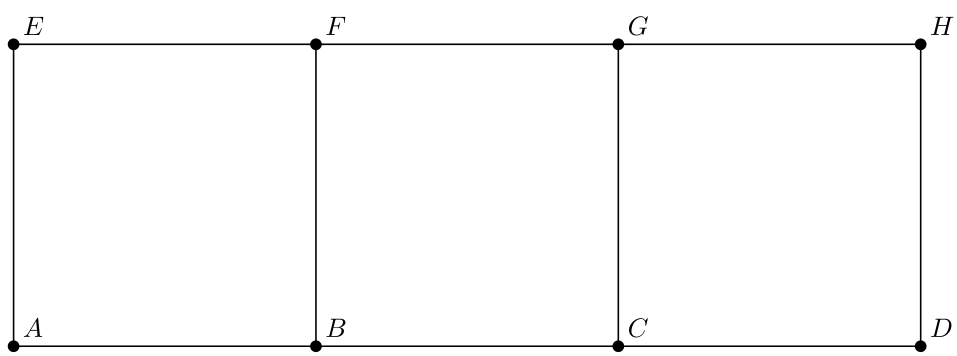
Solution: H spans G because G is connected, so every point has some path to the node. H is connected by construction. H is simple because G is simple (subgraphs of simple graphs are simple, basically by definition), and finally H is a tree because by construction there is only path between any two vertices of H, so there is no cycle which contains any two vertices of H, and therefore no cycles in H.

Along with depth-first search, there is one more common algorithm to find spanning trees.

Definition 3 *The **breadth-first search** algorithm for a connected simple graph G is given by*

1. *Choose a vertex of G, called the node, and add it to a subgraph H.*
2. *Starting at the node, add all edges coming out of the node and all vertices these connect to to H. Call these vertices the level 1 vertices.*
3. *Take a level 1 vertex and take all edges coming out of it which connect to vertices not previously added to H. Add these edges and vertices to H, with these vertices being level 2 vertices.*
4. *Repeat Step 3 for each level 1 vertex.*
5. *Repeat Steps $3 - 4$ for all level 2 vertices, then all level 3 vertices, and so on, until all vertices of some level don't connect to any more unvisited vertices.*

Problem 4 *Use breadth-first search (with the node at A) to find a spanning tree of the following graph.*

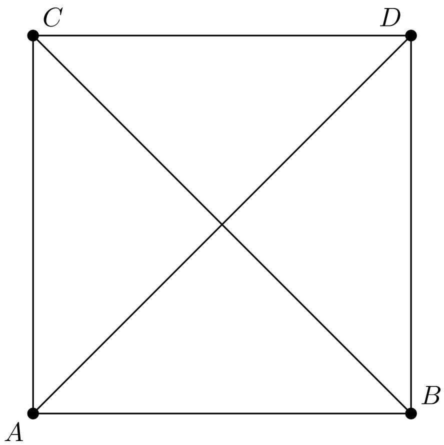


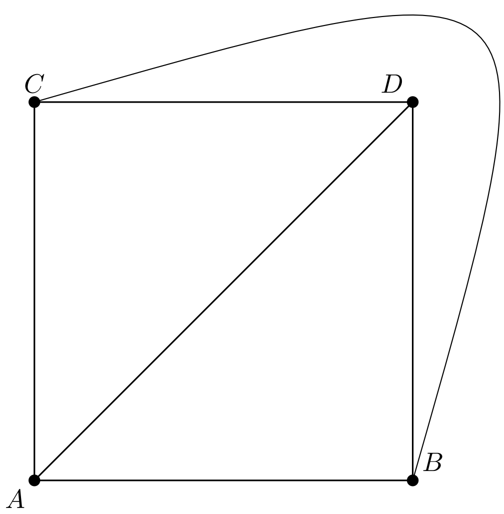
Solution: There are once again many answers - for instance, the answer will be different depending on whether B or E is the first level 1 vertex taken in Step 3. Students should notice the differences between their answer here and their answer to Problem 2.

2 Planar Graphs

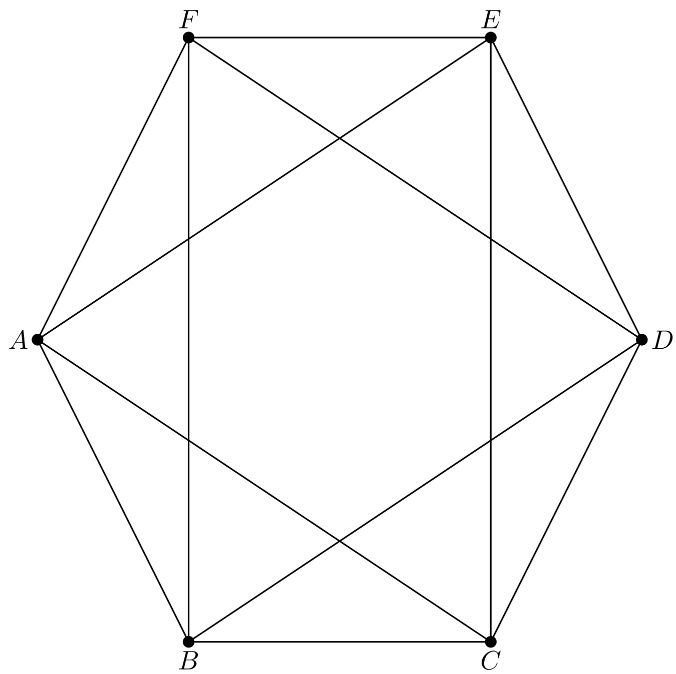
Definition 4 *A (connected, simple) graph G is **planar** if it can be drawn in the plane without edges intersecting (except at vertices).*

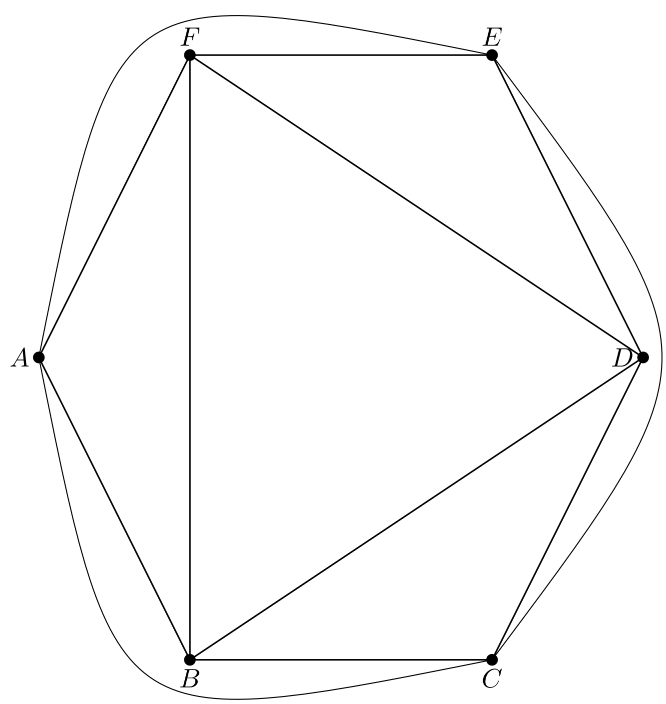
Problem 5 *Even though the following graphs are drawn with intersecting edges, they are still planar because they can be drawn without them. For each graph, show that it's planar by drawing it without intersecting edges.*





Solution:





Solution:

Problem 6 *Show that every subgraph of a planar graph is planar.*

Solution: Let G be a planar graph, and draw it without intersecting edges. Then any subgraph H cannot create intersecting edges because it can only use vertices and edges of G.

Problem 7 *Show that all trees are planar. (Hint: It may help to think of the breadth-first search algorithm.)*

Solution: Fix a node of the tree, and draw all of its branches (every edge coming out of it) far away from each other. For each level 1 vertex, draw all of its branches far away from each other as well. Repeating this breadth-first search, we cover all vertices and all edges, because breadth-first search creates a spanning tree, and the only spanning tree of a tree is itself.

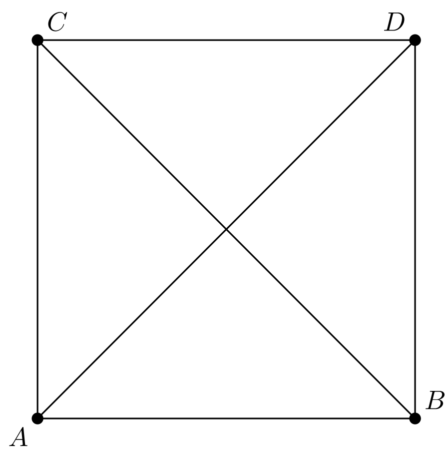
3 Euler Characteristic of Planar Graphs

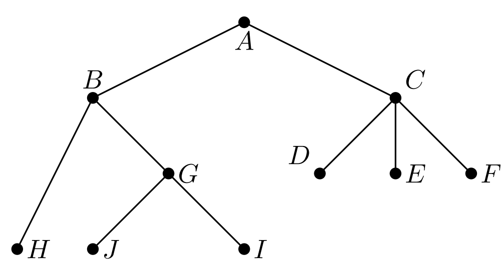
Definition 5 *Given a planar graph G drawn without intersecting edges, the regions that the edges of G divide the plane into are called the **faces** of G.*

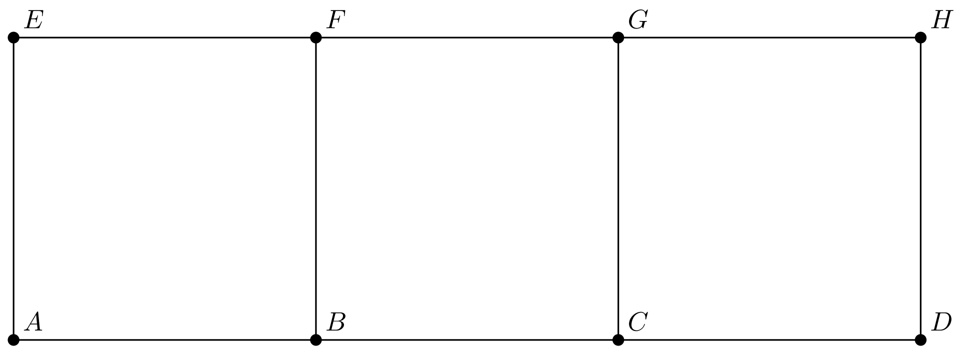
In addition to faces bounded by edges (which are called *interior faces*), every graph also has one *exterior face*, which can be thought of a face "at infinity", which contains everything that's far away from the graph.

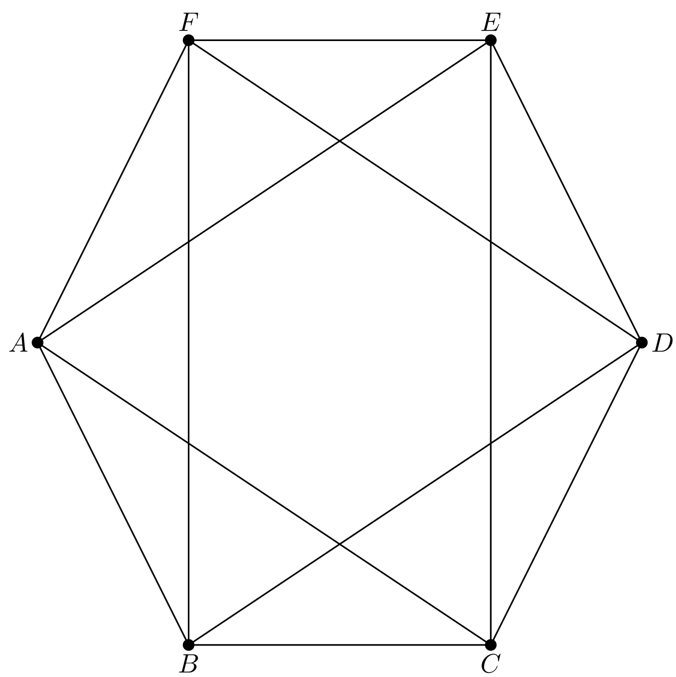
Definition 6 *Given a graph G with V vertices, E edges, and F, its **Euler characteristic** is given by $\chi(G) = V - E + F$.*

Problem 8 *For each of the following planar graphs, determine its Euler characteristic. (You may have to redraw some of them without intersecting edges.)*









Solution: They are all 2.

Theorem 1 *(Euler) The Euler characterstic of any planar graph is* 2*.*

Problem 9 *Let's prove Theorem* 1*.*

- How many faces does a tree have? Conclude that Theorem 1 holds in the case that G is a tree. (Hint: Refer to Problem 12 on last week's worksheet.)

  Solution: Because G has no cycles, it has no interior faces, so G has one face. By Problem 12 from last week, $E = V - 1$, so $V - E + F = V - (V - 1) + 1 = 2$.

- Let G be a graph and H be a subgraph which spans G. Suppose we add an edge of G into H. Show that this adds a cycle into H, and therefore adds one face to H.

  Solution: Suppose the new edge connects vertex x to vertex y. Because H spans G, H must already contain a path from x to y, so the new edge creates one new cycle, and therefore one new face.

- Conclude the proof of Theorem 1.

  Solution: Let H be a spanning tree of G. Then by the first part, Theorem 1 is true for H. One-by-one, add all the remaining edges of G into H. Since a spanning subgraph is still spanning if we add another edge, every edge we add by the second part will add one face, so Theorem 1 is still true at each step; in particular, it's true for G.

4 Bonus Section: Nonplanar graphs

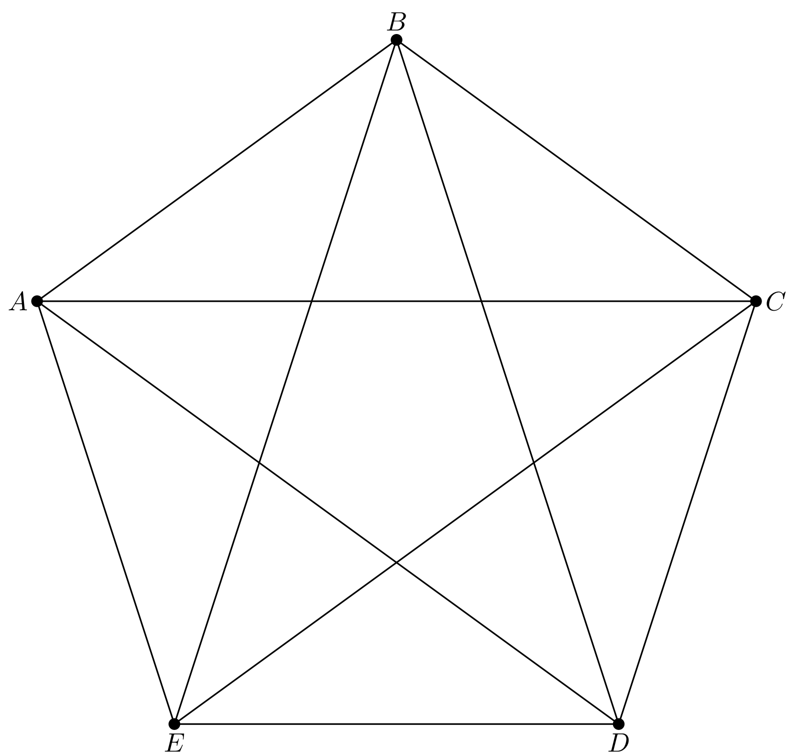
Using the Euler characteristic, we can finally show that certain graphs are *not* planar.

Problem 10 *Let G be a planar graph with more than 1 edge. Show that*

- $2E \geq 3F$
- $E \leq 3V - 6$

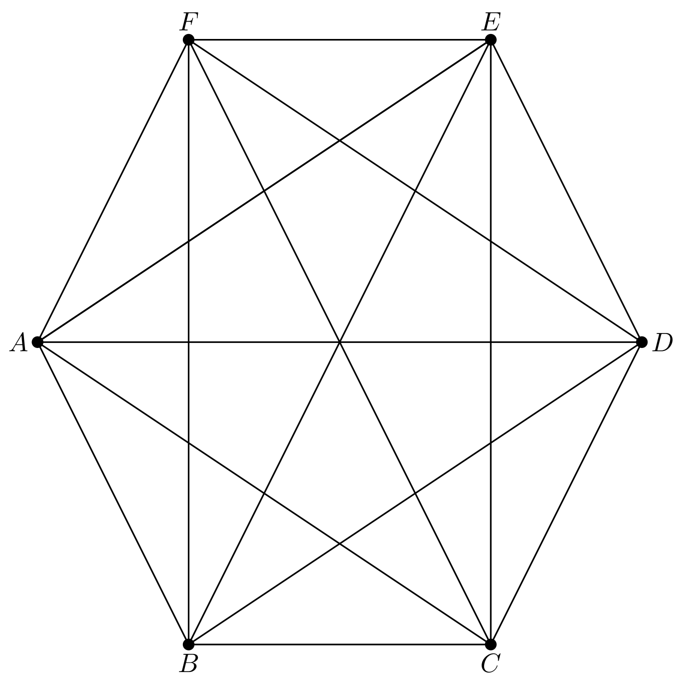
Solution: Because G has more than one edge, every face is bounded by at least three edges. But every edge is the intersection of exactly two faces, so the sum of "degrees" of the faces (i.e. the sum of the number of edges which bound each face) is equal to $2E$, and is also at least $3F$. Now by Theorem 1, $F = E - V + 2$, so $3(E - V + 2) \leq 2E$, and rearranging gives us $E \leq 3V - 6$.

Problem 11 *Using Problem 10, show that the following graph is not planar.*



Solution: This graph has 5 vertices and 10 edges, so by Problem 10 if it were planar then $10 \leq 3(5) - 6 = 9$, which is a contradiction, so it cannot be planar.

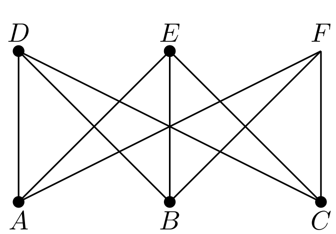
Problem 12 *Show that the following graph is not planar. (Hint: Suppose it were planar. Then all of its subgraphs would have to be planar. Can you find a subgraph which is not?)*



Problem 13 *Let G be a planar graph with more than* 1 *edge and no triangular faces. Show that* $E \leq 2V - 4$

Solution: Because G has more than one edge, every face is bounded by at least three edges, and therefore by at least 4 since there are no triangular faces. But every edge is the intersection of exactly two faces, so the sum of "degrees" of the faces (i.e. the sum of the number of edges which bound each face) is equal to $2E$, and is also at least $4F$. Now by Theorem 1, $F = E - V + 2$, so $4(E - V + 2) \leq 2E$, and rearranging gives us $E \leq 2V - 4$.

Problem 14 *Using Problem* 13*, show that the following graph is not planar.*



Solution: This graph has no triangular faces because it has no cycles with 3 edges, so if it were planar then by Problem 13 $9 \leq 2(6) - 4 = 8$, which is a contradiction.