Graph Theory I - Properties of Trees

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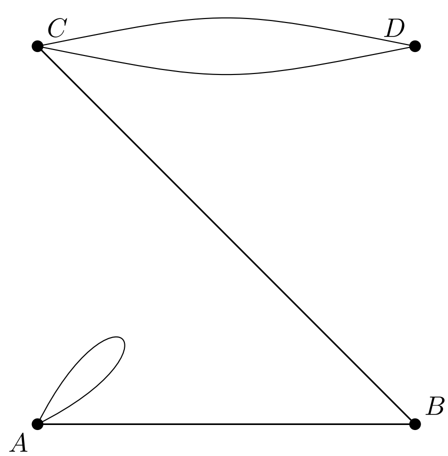
1 Graphs

Definition 1 *A* ***graph*** *G is a set $V(G)$ of points (called vertices) together with a set $E(G)$ of edges connecting the vertices.*

Though graphs are abstract objects, they are very naturally represented by diagrams, where we (usually) draw the vertices and edges in the plane. For instance, if we let G be the graph defined by $V(G) = \{A, B, C, D\}$ and $E(G)$ a set containing one edge connecting A and A, one connecting A and B, one connecting B and C, and two connecting C and D, we might represent it as follows.



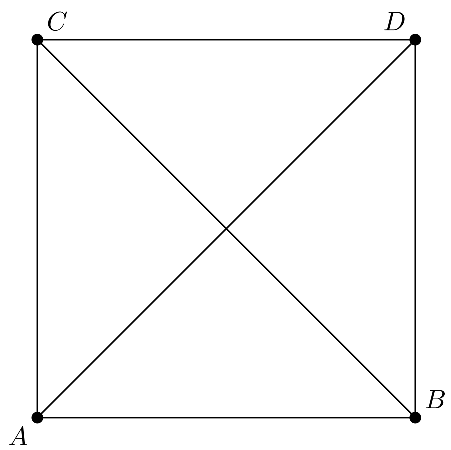
To eliminate a lot of bad behavior, we will also require some special properties of our graphs. As we'll see, almost all real-world applications of graphs do satisfy these properties, so it's not unreasonable to require them.

Definition 2

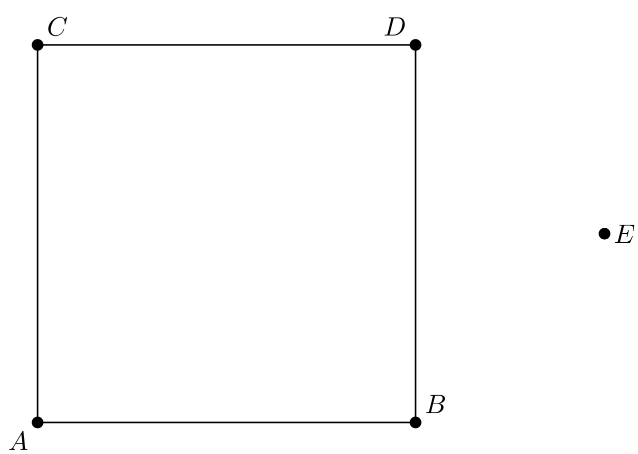
- *A graph is* ***connected*** *if every vertex is connected to some other vertex by an edge.*
- *A graph is* ***simple*** *if no vertex is connected to itself by an edge, and any two different vertices are connected by at most one edge.*

The graph above is *connected*, but is *not simple.*

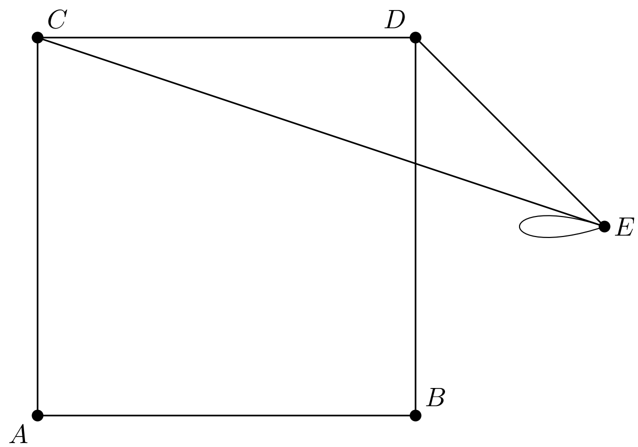
Problem 1 *For each graph below, decide whether it is connected, and also decide whether it is simple.*



Solution: This is connected and simple.



Solution: This is not connected (as E is not connected to any other vertex), but it is simple.



Solution: This is connected, but not simple (as E is connected to itself).

Problem 2 *For each of the following graphs, which is given to you by describing the sets of vertices and edges, determine whether it is connected, and also determine whether it is simple. Try to do this without drawing the graph if you can.*

- $V(G) = \{A, B, C, D\}$, $E(G)$ contains one edge connecting A and B, one connecting B and D, and one edge connecting A and D.

  Solution: G is not connected, since C is not connected to anything. But G is simple.

- $V(G) = \{A, B, C, D, E\}$, and $E(G)$ contains one edge connecting every pair of vertices.

  Solution: G is connected, since everything is connected to everything else, so in particular they are connected to something else. But G is not simple, because G also contains edges connecting any vertex to itself.

- $V(G) = \{A, B, C, D, E\}$, and $E(G)$ contains one edge connecting every pair of *different* vertices.

  Solution: Similarly to the previous part, G is connected, but G is simple this time, since the self-connections have been removed.

As previously mentioned, we can represent many real-world situations by graphs. For instance, if we have a group of people, some of whom are friends with each other, we can represent this as a graph by letting our vertices be the people, and connecting two people with an edge if they are friends. Assume that everyone has at least one friend in this group.

Problem 3 *Is this graph connected? Is it simple?*

Solution: Because we assumed everyone has at least one friend in this group, every vertex is connected to some other vertex, so the graph is connected. Because two people can't be friends more than once, they can't be connected more than once, and because no one is friends with themselves, this means the graph is also simple.

Problem 4 *For each example below, determine if the resulting graph is connected, and also determine whether it is simple.*

- A family tree, where people are related if one is the other's parent/child.

- A set of bus stations, where stations are related if there is a bus which goes directly from one to the other.

- A molecule, where atoms are related if they're bonded.

- A contact tracing network, where two people are related if they've been in close enough contact to spread an infectious disease.

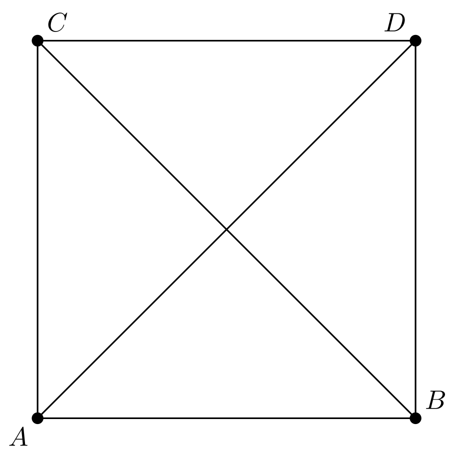
Solution: All of these are examples of connected graphs. The first and last are simple, and so are the second and third if you interpret it in the right way.

2 Degrees of Vertices

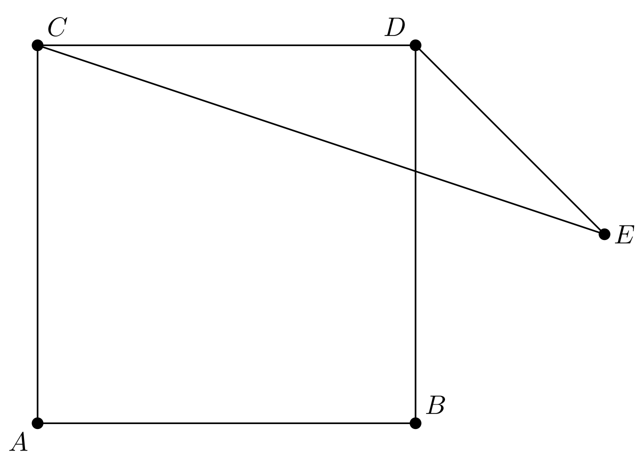
From now on, *all graphs are connected and simple.*

Definition 3 *The* ***degree*** *of a vertex* $x \in V(G)$ *is the number of other vertices connected to it.*

Problem 5 *For the following graphs, label the degree of each vertex.*



Solution: All vertices have degree 3.



Solution: Vertices A, B, and E have degree 2, while vertices C and D have degree 3.

Problem 6 *Using the fact that G is simple, show that the degree of a vertex x is the same as the number of edges coming out of it.*

Solution: Since G is simple, each edge coming out of x connects to a different vertex, and no two of these edges can connect to the same vertex either. For any other vertex connected to x, by definition there is an edge coming out of x connecting to it. In other words, there is a bijection between the set of vertices connected to x and the set of edges coming out of x.

Problem 7 *Show that the sum of the degrees of every vertex of G is even.*

Solution: Because G is simple, when we add up all the degrees, every edge gets counted exactly twice, so since there are an integer number of edges, the sum of the degrees is even.

Problem 8 *Use the previous problem to show that in a group of seven people, not everyone can have exactly three friends.*

Solution: If this were the case, then the sum of the degrees would be 21, which is odd.

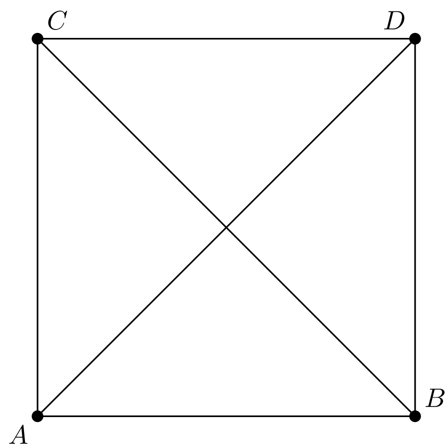
3 Trees

Definition 4 *Given a graph G,*

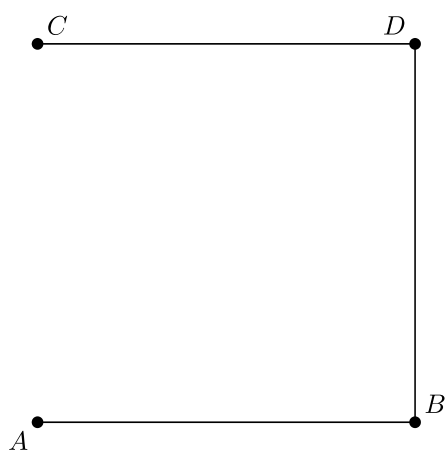
- *A **path** in G is a sequence of edges such that each edge begins where the previous edge ends and ends where the next edge begins.*
- *A **cycle** in G is a path starting and ending at the same vertex.*

*G is called a **tree** if it contains no cycles.*

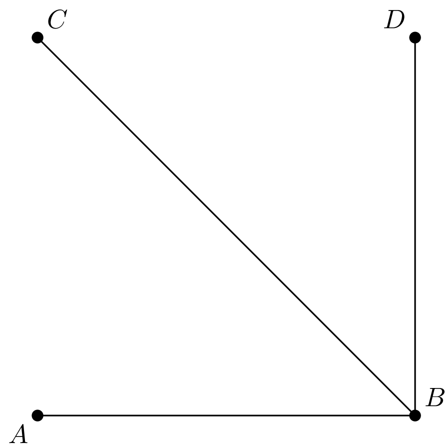
Problem 9 *For each of the following graphs, determine whether or not it is a tree.*



Solution: This is not a tree; for instance, it has a cycle going from A to B to C back to A.



Solution: This is a tree.



Solution: This is a tree.

Problem 10 *Show that every family tree is a tree.*

Solution: A cycle in a family tree means that there is a sequence of different people who are all parents/children of the next person. Students should realize why this is not possible.

Problem 11 *Show that every tree with an edge has a vertex of degree* 1*. Such a vertex is called a* ***leaf*** *of the tree. (Hint: Consider the longest path in G, and take one of its endpoints x. x has to be connected to one other vertex because it's on this path, so show that it can't be connected to any other vertices.)*

Solution: Let P be the longest path in G, and let x be one of its endpoints. x is connected to the next vertex along P, but if x is also connected to any other vertex y along P, then that would make a cycle from x to y (along P) and then back to x. If x is connected to any vertex outside P, then we could make P longer by adding that edge to the end, which contradicts that P is the longest path. Therefore x can only be connected to that one vertex, so its degree is 1.

Problem 12 *Let G be a tree, and let V be the number of its vertices, and E the number of its edges. Let us prove that*

$$E = V - 1$$

We will prove this by *induction on the number of vertices.*

- The base case is $V = 1$. What is the only thing a connected simple graph with one vertex can be? Check that the base case is satisfied.

  Solution: By simplicity, a graph with one vertex cannot have any edges. This satisfies the base case, because this is a tree.

- State the inductive hypothesis. (Be extremely careful with your wording!)

  Solution: Suppose that for some positive integer n that **all** trees with n vertices have $n - 1$ edges. (Note: It's important that you assume this for all trees, because you're proving this statement for all trees!)

- Finish the induction. (Hint: By Problem 11 we can find a leaf. What can we say about the leaf and about the rest of the tree?)

  Solution: Since n is a positive integer, $n + 1 \geq 2$ and so G has an edge. By Problem 11, let x be a leaf of G. Then by Problem 6, there is only one edge coming out of x. The rest of the tree, apart from the leaf and this one edge, is still a tree, and has n vertices, so by the inductive hypothesis the rest of the tree has $n - 1$ edges. But then G has n edges, which completes the induction.

4 Bonus Section: Hamiltonian Paths

Definition 5 *A path in a graph G is called **Hamiltonian** if it visits each vertex exactly once.*

One interesting case occurs if we look at a map of the United States. We can it into a graph by letting each state be a vertex, and connecting them if they share a border (corners, such as Utah and New Mexico, do not count). By excluding Alaska and Hawaii, we obtain a connected graph, and of course this is simple (check this!).

Problem 13 *Show that there is no Hamiltonian path on this graph which starts or ends at New York.*

Solution: None of the New England states connects to any of the states west of New York, so suppose there were a Hamiltonian path starting at New York. It would have go either east, in which case it could never leave New England since it can never go to New York again, or west, in which case it could never enter New England since it can never go to New York again. The logic for not ending at New York is similar.

Problem 14 *Suppose a Hamiltonian path starts in New England. Apart from New York, there is one other state where it cannot end in. Find it.*

Solution: Georgia. It's adjacent to South Carolina and Florida, both of which have degree 2, so suppose we tried to end in Georgia. Then either South Carolina or Florida would have to have only one connection in our path, since both of them can't be connected to Georgia if we end there. But any state in the middle of our path has to have two connections, so we get a contradiction.