Graph Theory II - Planar Graphs

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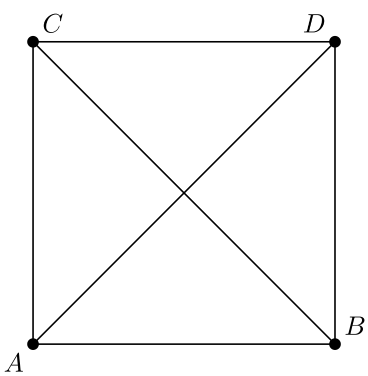
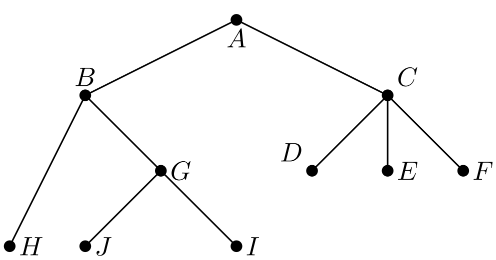
1 Spanning Trees

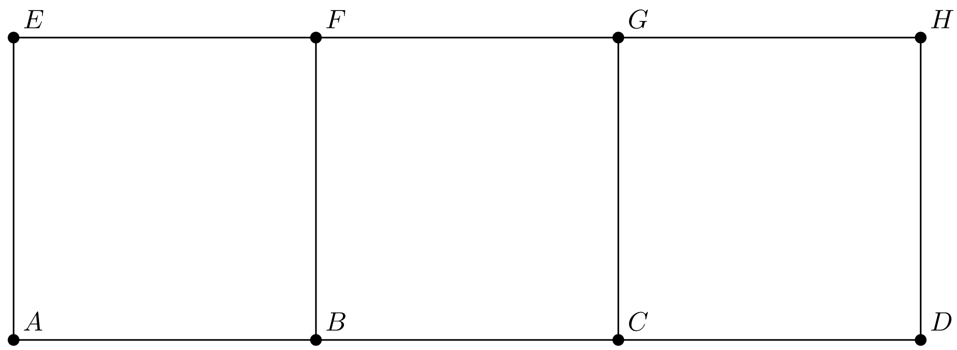
Recall from last week that a tree is a connected simple graph with no cycles.

Definition 1 *Given a graph G, a* ***subgraph*** *of G is a graph H such that $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. H is said to* ***span*** *G if H contains all the vertices of G. H is a* ***spanning tree*** *of G if it spans G and is also a tree.*

Problem 1 *For each graph below, draw one spanning tree (there may be more than one!)*





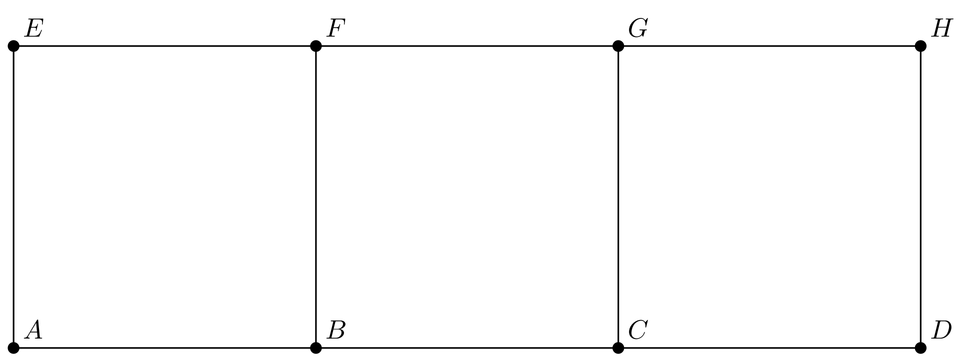


As one may begin to suspect, it is in fact the case that all graphs have spanning trees. To prove this, it suffices to provide an algorithm which gives a spanning tree.

Definition 2 *The **depth-first search** algorithm for a connected simple graph G is given by*

1. *Choose a vertex of G, called the node, and add it to a subgraph H.*
2. *Starting at a vertex, travel along an edge to any vertex not previously visited. Add both the edge and the vertex to H.*
3. *Repeat Step 2 until a vertex is reached whose neighbors have all been previously visited. Return to the node when this occurs.*
4. *Repeat Steps 2 and 3 until all neighbors of the node have been visited.*

Problem 2 *Use depth-first search (with the node at A) to find a spanning tree of the following graph.*



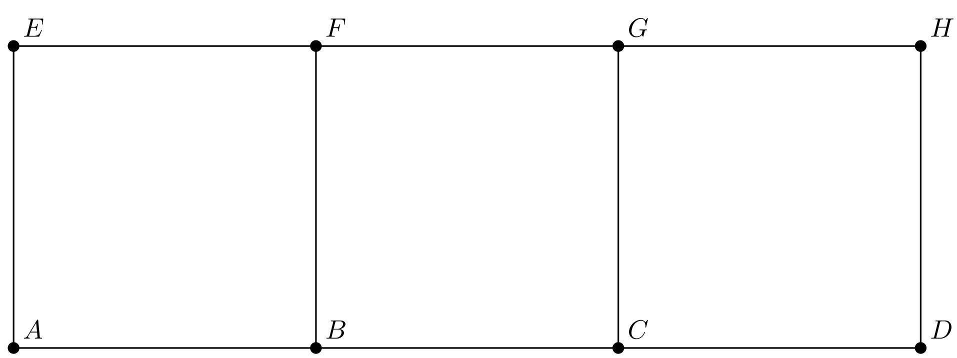
Problem 3 *Explain why depth-first search always results in a spanning tree H.*

Along with depth-first search, there is one more common algorithm to find spanning trees.

Definition 3 *The* ***breadth-first search*** *algorithm for a connected simple graph G is given by*

1. *Choose a vertex of G, called the node, and add it to a subgraph H.*
2. *Starting at the node, add all edges coming out of the node and all vertices these connect to to H. Call these vertices the level 1 vertices.*
3. *Take a level 1 vertex and take all edges coming out of it which connect to vertices not previously added to H. Add these edges and vertices to H, with these vertices being level 2 vertices.*
4. *Repeat Step 3 for each level 1 vertex.*
5. *Repeat Steps $3-4$ for all level 2 vertices, then all level 3 vertices, and so on, until all vertices of some level don't connect to any more unvisited vertices.*

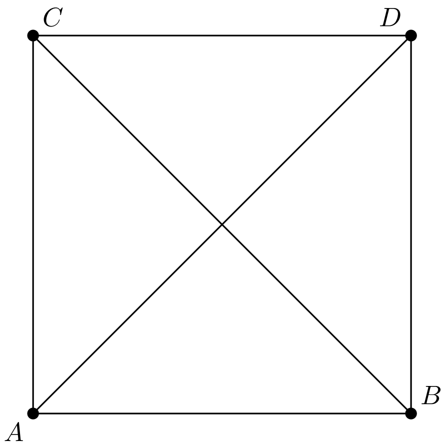
Problem 4 *Use breadth-first search (with the node at A) to find a spanning tree of the following graph.*

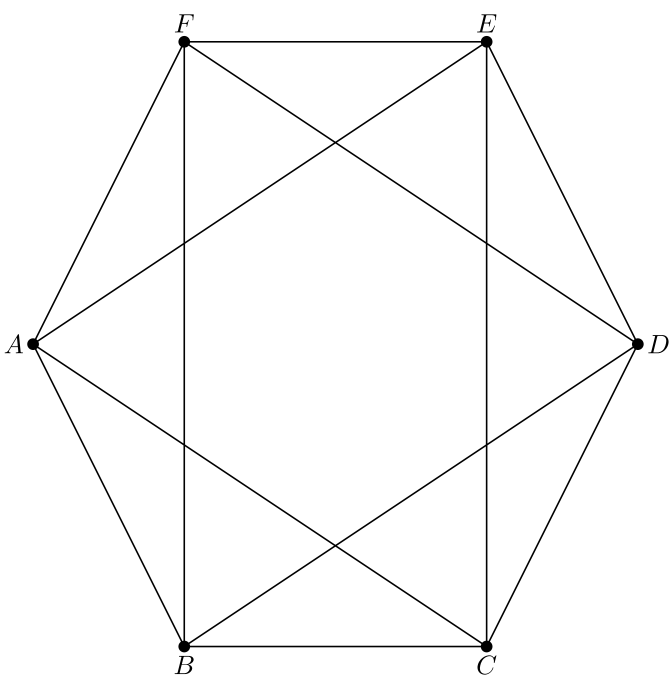


2 Planar Graphs

Definition 4 *A (connected, simple) graph G is* ***planar*** *if it can be drawn in the plane without edges intersecting (except at vertices).*

Problem 5 *Even though the following graphs are drawn with intersecting edges, they are still planar because they can be drawn without them. For each graph, show that it's planar by drawing it without intersecting edges.*





Problem 6 *Show that every subgraph of a planar graph is planar.*

Problem 7 *Show that all trees are planar. (Hint: It may help to think of the breadth-first search algorithm.)*

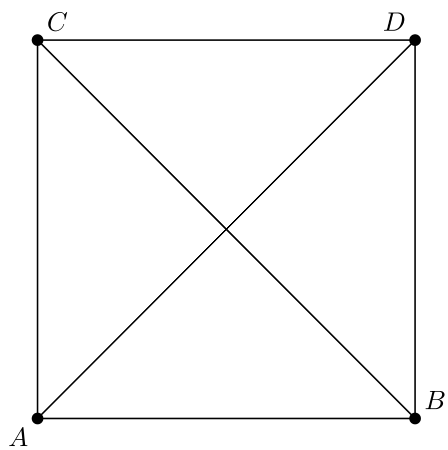
3 Euler Characteristic of Planar Graphs

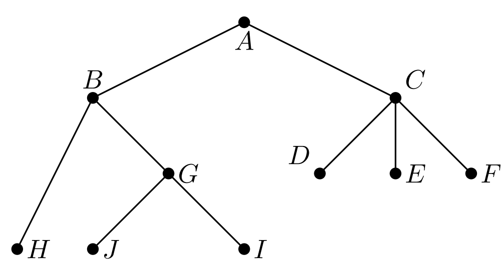
Definition 5 *Given a planar graph G drawn without intersecting edges, the regions that the edges of G divide the plane into are called the **faces** of G.*

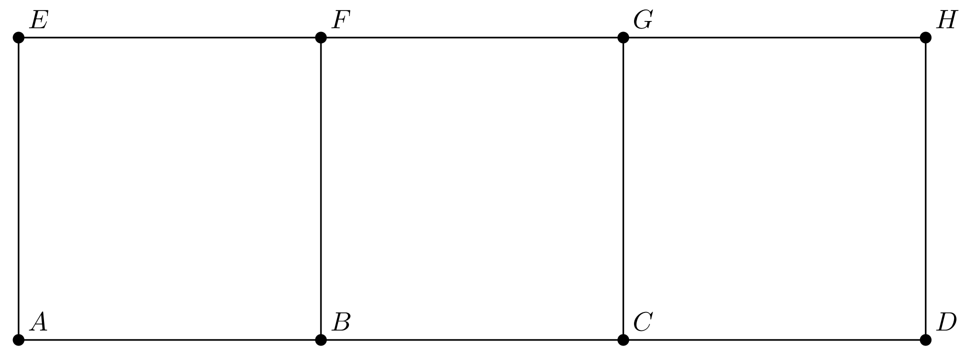
In addition to faces bounded by edges (which are called *interior faces*), every graph also has one *exterior face*, which can be thought of a face "at infinity", which contains everything that's far away from the graph.

Definition 6 *Given a graph G with V vertices, E edges, and F, its **Euler characteristic** is given by $\chi(G) = V - E + F$.*

Problem 8 *For each of the following planar graphs, determine its Euler characteristic. (You may have to redraw some of them without intersecting edges.)*






E
F
G
H
A
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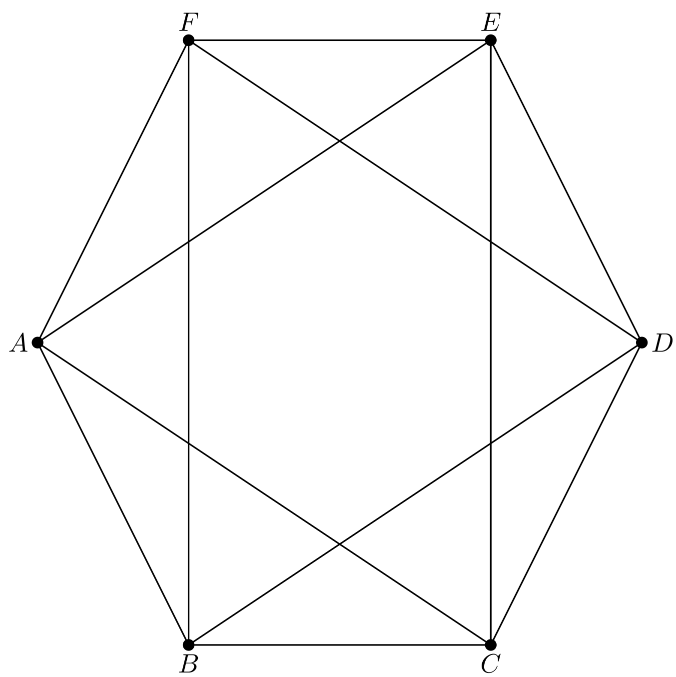
C
D


F
E
A
D
B
C

Theorem 1 *(Euler) The Euler characterstic of any planar graph is* 2*.*

Problem 9 *Let's prove Theorem* 1*.*

- How many faces does a tree have? Conclude that Theorem 1 holds in the case that G is a tree. (Hint: Refer to Problem 12 on last week's worksheet.)

- Let G be a graph and H be a subgraph which spans G. Suppose we add an edge of G into H. Show that this adds a cycle into H, and therefore adds one face to H.

- Conclude the proof of Theorem 1.

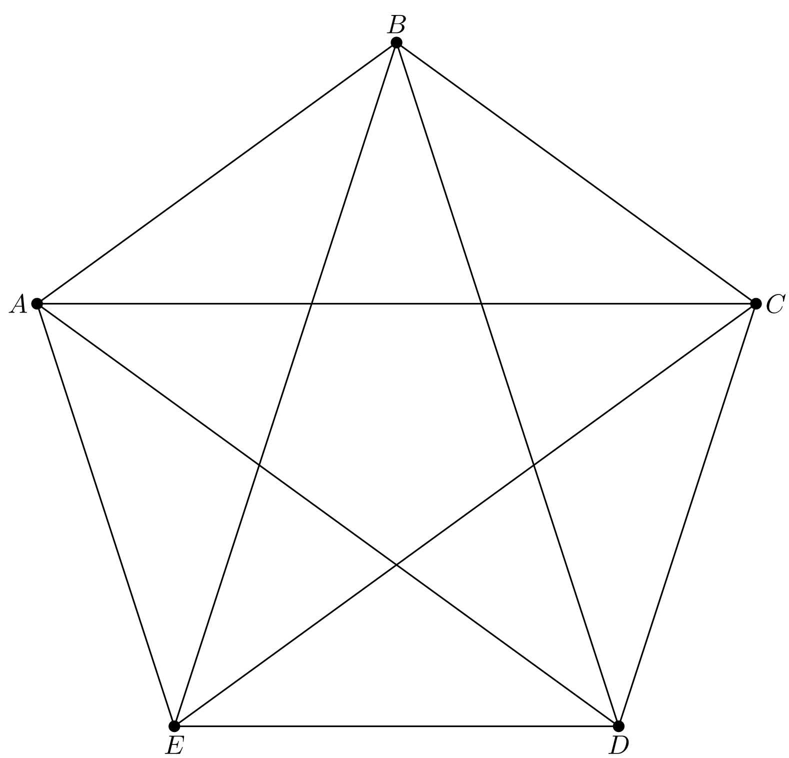
4 Bonus Section: Nonplanar graphs

Using the Euler characteristic, we can finally show that certain graphs are *not* planar.

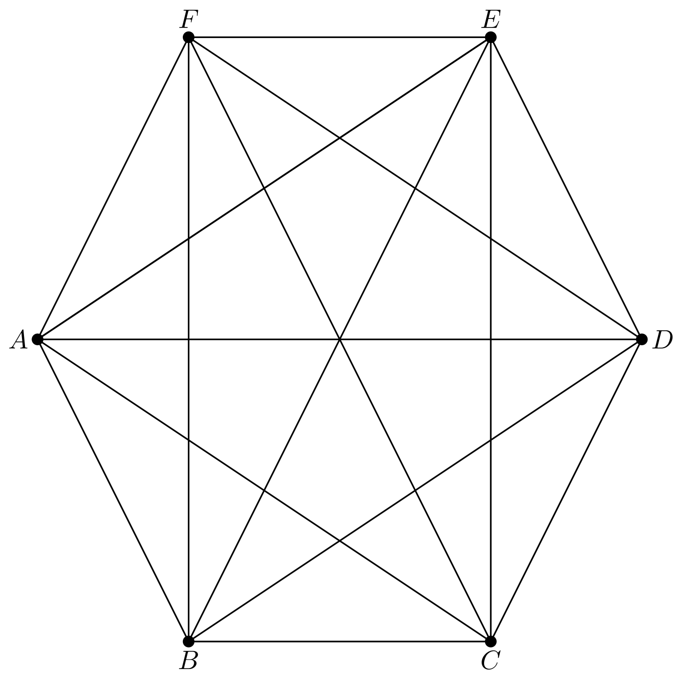
Problem 10 *Let G be a planar graph with more than* 1 *edge. Show that*

- $2E \geq 3F$
- $E \leq 3V - 6$

Problem 11 *Using Problem* 10*, show that the following graph is not planar.*



Problem 12 *Show that the following graph is not planar. (Hint: Suppose it were planar. Then all of its subgraphs would have to be planar. Can you find a subgraph which is not?)*



Problem 13 *Let G be a planar graph with more than 1 edge and no triangular faces. Show that $E \leq 2V - 4$*

Problem 14 *Using Problem 13, show that the following graph is not planar.*