ORMC Olympiad Group
Winter: Week 4
Euler's Phi Function, and Remaining Problems

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Problems

1. (Santos, p.27) Find the units digit of 7^{7^7}. Here, and the upcoming problems, by a^{b^c}, we mean $a^{(b^c)}$.
2. Find $2^{50^{50}}$ in mod 13.
3. Find the last two digit of $12^{11^{10}}$
4. In the last class we learned **Euler's Thm** and **Euler's Phi Function**. Recall, for $n > 1$, $\phi(n)$ is define to be the number of integers among the set $\{1, 2, 3, \ldots, n-1\}$ that are relatively prime with n. For example, $\phi(5) = 4, \phi(6) = 2$
5. Show that for $n = p^\alpha$, where p is a prime number, there are exactly $\varphi(p^\alpha) = (p-1)p^{\alpha-1}$ positive integers less than n which relatively prime to p^α.
6. Show that for $n = pq$, where p and q are different prime numbers, there are exactly $\varphi(pq) = (p-1)(q-1)$ positive integers less than n which are relatively prime to pq.
7. Show that for $n = p^2q$, where p and q are different prime numbers, there are exactly $\varphi(p^2q) = (p^2 - p)(q-1)$ positive integers less than n

which are relatively prime to $n = p^2q$.
HINT: Remember we did this problem for $n = p^\alpha$ and $n = pq$. This is just a simple generalization.

8. Show that for $n = p^\alpha q^\beta$, where p and q are different prime numbers, and $\alpha, \beta \geq 1$ integers, there are exactly $\varphi(n) = (p^\alpha - p^{\alpha-1})(q^\beta - q^{\beta-1})$ positive integers less than n which are relatively prime with n.

9. Find all positive integers (n, m, k) such that $k \geq 2$ such that
$$1! + 2! + \cdots + n! = m^k$$

10. **(JBMO-2007)** Let p be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.

11. A number m in *mod* n is called perfect square if there exist an integer x such that $m \equiv x^2 \mod n$. For example, $0, 1, 4$ are perfect squares in *mod* 5, and $0, 1, 2, 4$ are perfect squares in *mod* 7. Find the total number of perfect squares in mod 25.

12. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 1, \ldots, a_{n-1} + n - 1$ is also a complete residue class?

13. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 3, a_2 + 6, \ldots, a_{n-1} + 3(n-1)$ is also a complete residue class?

14. Let $p > 2$ be a prime number. Is there a complete residue class $\{a_1, \ldots, a_{p-1}\} = \{1, 2, 3, \ldots, p-1\}$ in $\mod p$ so that $a_1, 2a_2, 3a_3, \ldots, (p-1)a_{p-1}$ is also a complete residue class?
HINT: Consider the product and use Wilson's theorem

15. Find $2^{1000^{1000}}$ in mod 7.

16. Find $2^{1000^{1000}}$ in mod 15.

17. Find the last three digits of the number 2013^{1000}.

18. Find the last three digits of the number 2017^{2017}.

19. (Canada MO 2003) Find the last three digits of the number $2003^{2002^{2001}}$.

20. Find $\sum_{d|n} \phi(d)$ in terms of n. Prove it.