ORMC Olympiad Group 2022
Winter Week 2 Solutions

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Solution 1

Consider the following $n-1$ numbers modulo n:

$$a, 2a, \ldots, (n-1)a.$$

If some two of these are equal, say, $ia \equiv ja \pmod{n}$, then $a(i-j) \equiv 0 \pmod{n}$. Since $\gcd(a,n)=1$, this implies $i-j \equiv 0 \pmod{n}$. However, i, j are distinct integers from 1 to $n-1$, so this is not possible. Hence the $n-1$ numbers in the set above are distinct modulo n.

These are all also clearly numbers from 1 to $n-1$ when taken modulo n (i.e. cannot be 0), so they are in fact $1, \ldots, n-1$ in some order. Thus, one of them must be b. That is, there exists some x such that $ax \equiv b \pmod{n}$. □

Solution 2

It suffices to find x such that $ax \equiv 1 \pmod{b}$. By the result of the previous problem, this is possible. □

Solution 3

For $a \equiv 0 \pmod{p}$, it is obvious. Now, consider a with $\gcd(a,p)=1$. Consider the set of the $p-1$ numbers

$$\{a, 2a, \ldots, (p-1)a\}.$$

If some $ia \equiv ja \pmod{p}$ for $i \neq j$, then we have $a(i-j) \equiv 0 \pmod{p}$. But since i, j are distinct and from 1 to $p-1$, clearly $i-j \not\equiv 0 \pmod{p}$. So this would contradict the fact that $\gcd(a,p)=1$. Thus, the above set consists of $p-1$ distinct numbers modulo p. These must in fact be $1, 2, \ldots, p-1$ in some order. So,

$$(a)(2a)(\cdots)((p-1)a) \equiv (1)(2)(\cdots)(p-1) \pmod{p} \implies a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}.$$

Since clearly $\gcd(p,(p-1)!)=1$, we can then conclude $a^{p-1} \equiv 1 \pmod{p}$, as desired. □

Solution 4

Note that by sum of cubes factorization,

$$n^3 + 100 = n^3 + 1000 - 900 = (n+10)(n^2 - 10n + 10) - 900.$$

In particular, $n^3 + 100 \equiv -900 \pmod{(n+10)}$. This means $n + 10 \mid 900$. Then it is clear that the largest possible value of n is $\boxed{890}$. □

Solution 5

The given expression is similar to $(n-10)^3 = n^3 - 30n^2 + 300n - 100$; in fact, note that

$$n^3 - 30n^2 + 200 \equiv n^3 - 30n^2 + 300n + 200 - 300 \equiv (n-10)^3 \pmod{300}.$$

So, $300 \mid (n-10)^3$. This does not imply that $300 \mid (n-10)$, since 300 is not prime. Instead, since $300 = 2^2 \cdot 3 \cdot 5$, note that $2^2 \mid (n-10)^3 \implies 2 \mid (n-10)$, and so on; thus, $2 \cdot 3 \cdot 5 \mid (n-10)$ is the equivalent condition. Then there are $\boxed{17}$ possible n, i.e. $30(0)+10, \ldots, 30(16)+10$. □

Solution 6

The difference of the two expressions must also be divisible by p; that is, $p \mid 2n+1$, so $2n \equiv -1 \pmod{p}$. By the first given condition, we have

$$n^2 \equiv -3 \pmod{p} \implies 4n^2 \equiv -12 \pmod{p}.$$

But then note that $4n^2 \equiv (2n)^2 \equiv (-1)^2 \pmod{p}$, so

$$1 \equiv -12 \pmod{p} \implies 13 \equiv 0 \pmod{p}.$$

Thus $p \mid 13 \implies p = 13$ is the only solution. Indeed, $n = 6$ works with this p, so $p = \boxed{13}$. □

Solution 7

First we will consider a general term of the form $\binom{p-3}{k}$ where p is a prime. Note that we can write this as

$$\begin{aligned} \frac{(p-3)!}{k!(p-3-k)!} &= \frac{(1)(2)(\cdots)(p-3)}{(1)(2)(\cdots)(k)(1)(2)(\cdots)(p-3-k)} \\ &\equiv \frac{(k+1)(\cdots)(p-3)}{(p-1)(p-2)(\cdots)(k+3)(-1)^{p-3-k}} \\ &\equiv \frac{(k+1)(k+2)}{(-1)(-2)(-1)^{p-3-k}} \pmod{p}. \end{aligned}$$

The simplifications made were cancelling terms (which is allowed since p is prime, and thus relatively prime to all of the positive integers less than it) and converting i to $-(p-i)$. Then, since p is odd this is just $\dfrac{(k+1)(k+2)(-1)^k}{2} \pmod{p}$. So, now we must simply evaluate

$$\sum_{k=0}^{62} \frac{(k+1)(k+2)(-1)^k}{2} \pmod{2017}.$$

Now, note that $\frac{(k+1)(k+2)}{2} = 1 + 2 + \ldots + (k+1)$. Thus, for odd k,

$$\frac{(k+1)(k+2)(-1)^k}{2} + \frac{(k+2)(k+3)(-1)^{k+1}}{2} = -(1+2+\ldots+k+1) + (1+2+\ldots+k+2) = k+2.$$

Hence the sum is simply the sum of odd numbers

$$1 + 3 + 5 + \ldots + 61 = 32^2 \equiv \boxed{1024} \pmod{17}.$$

□

Solution 8

Note that $111 = 3 \cdot 37$. In particular, this gives that $37 \mid 999$, so $1000 \equiv 1 \pmod{37}$. This motivates splitting the number into three digit numbers and summing those. If we let the number be $\overline{a231b312c}$, then it is equivalent to

$$\overline{a23} + \overline{1b3} + \overline{12c} \equiv (100a + 10b + c) + 246 \pmod{37}.$$

This must be $0 \pmod{37}$, so simplifying gives $\overline{abc} \equiv 13 \pmod{37}$. So we must find the number of three digit numbers equivalent to 13 modulo 37, and each of these will correspond to one choice of the digits a, b, c. These numbers are $37(3) + 13, \ldots, 37(26) + 13$, for a total of $\boxed{24}$ numbers. □

Solution 9

Note that $100 \equiv 1 \pmod{99}$; thus any $100^k \equiv 1 \pmod{99}$. So we can split the number into two digit numbers and then add those, and the result will be equivalent to the original number. Thus, we want the smallest two digit n such that $12+13+\ldots+n \equiv 0 \pmod{99}$. Equivalently, this is the sum of 1 to n minus the sum of 1 to 11, so

$$\frac{n(n+1)-11(12)}{2} \equiv 0 \pmod{99} \iff n(n+1)-11(12) \equiv 0 \pmod{99}.$$

(This is true since $\gcd(2,99)=1$). The left side factors, so we have $(n-11)(n+12) \equiv 0 \pmod{99}$. Now note that $99 = 9 \cdot 11$, and since 11 is prime this implies n is either 0 or -1 modulo 11. Testing these values (they must also be either 2 or 6 modulo 9) starting with 21 gives that $n = \boxed{33}$ is the smallest that works. □

Solution 10

Note that $p=2$ does not work, and that $p=3$ does work. Now for $p>3$, we consider the numbers modulo 3. Clearly $p \not\equiv 1 \pmod{3}$. If $p \equiv 1 \pmod{3}$, then $p^2+p+1 \equiv 0 \pmod{3}$ and thus cannot be prime. But if $p \equiv 2 \pmod{3}$, then $p+10 \equiv 0 \pmod{3}$ and thus cannot be prime. So, no $p>3$ can work. Thus the answer is $\boxed{3}$. □

Solution 11

Let the n numbers be $a_1, \ldots, a_n$. Then, consider the n sums

$$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad \ldots \qquad , s_n = a_1 + a_2 + \ldots + a_n.$$

That is, s_i is the sum of a_1 through a_i. When taken modulo n, these are all numbers from 0 to $n-1$. If any one of them is 0, then we are done. If none of them are 0, then this is a set of n numbers between 1 and $n-1$. Thus (by pigeonhole principle) there must exist some two of them equal; that is, $s_i = s_j$ for some $i<j$. Then note that $s_j - s_i \equiv a_{i+1} + \ldots + a_j \equiv 0 \pmod{p}$, as desired. □

Solution 12

Note that 109 is prime. Then, by FLT (Fermat's little thm), $k^{108} \equiv k \pmod{109}$ for each $k = 2, 3, 6$. Hence $k^{107} \equiv \frac{1}{k} \pmod{109}$. Then,

$$2^{107} + 3^{107} + 6^{107} \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \equiv \boxed{1} \pmod{109}.$$

□

Solution 13

The answer is $\boxed{\text{no}}$. Consider any number consisting of these numbers in some order. Then it will be equal to $\sum_{i=1}^{2008} (i^i)(10^{a_i})$ where the $a_i \geq 0$, i.e. the sum of the i^i times some power of 10. Now take this expression modulo 3. Since $10 \equiv 1 \pmod{3}$, we can ignore the powers of 10, so it is equivalent to $\sum_{i=1}^{2008} i^i$.

Now note that any $a^3 \equiv a \pmod 3$; in particular, this means $a^k \equiv a \pmod 3$ for any odd k, and $a^k \equiv a^2 \pmod 3$ for even k. Hence the expression is

$$\sum_{i=1}^{1004}(2i-1) + \sum_{i=1}^{1004}(2i)^2.$$

The first term is the sum of the first 1004 odd integers and thus equals 1004^2. For the second term, take out the factor of 4 from each term and then use sum of squares; so, the expression is

$$1004^2 + (4)\frac{(1004)(1005)(2009)}{6} \equiv 2^2 + 2(1004)(335)(2009) \equiv 1 + 2(2)(2)(2) \equiv 2 \pmod 3.$$

But no square can be $2 \pmod 3$, so this expression can never be a perfect square. □

Solution 14

Consider the equation modulo 81. For $n \geq 9$, we have that $81 \mid n!$. Also, note that by computation,

$$\sum_{i=1}^{8} i! \equiv 63 \pmod{81}.$$

In particular, this means that for $n = 8$, the left side of the equation is $81j + 63$, and as noted above, this will in fact still be some $81j + 63$ for any $n \geq 8$ because we will only add multiples of 81.

But this means that for $n \geq 8$, the equation is $81j + 63 = m^k \implies 9(9j + 7) = m^k$. Note that the highest power of 3 dividing the left side is 3^2, which implies $k \leq 2$ (since $3 \mid m^k \implies 3 \mid m$, so then $k \leq 2$). It is given that $k \geq 2$, so $k = 2$.

Then, the solutions (see problem 5) are $(n, m) = (3, 3), (1, 1)$, so the answer is $\boxed{(3, 3, 2), (1, 1, k), k \geq 2}$ (we can easily verify that there exist no solutions with other k for $n \leq 7$). □

Solution 15

Will be posted later classes.