ORMC Olympiad Group 2022
Winter Week 1 Solutions

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Solution 1

First consider the k^3 term. First we compute $k \pmod{10}$. Note that the units digit of powers of 3 repeats in cycles of length 4: $3, 9, 7, 1, \ldots$. Thus,

$$k = 2022^3 + 3^{2022} \equiv 2^3 + 3^2 \equiv 7 \pmod{10}$$

(Note the simplification $3^{2022} \equiv 3^2 \pmod{10}$. Thus, $k^3 \equiv 7^3 \equiv 3 \pmod{10}$.

Now, consider the 3^k term. As noted above, powers of 3 cycle with a length of 4. That is, if $k \equiv j \pmod{4}$, then $3^k \equiv 3^j \pmod{10}$. So first we must compute k modulo 4:

$$2022^3 + 3^{2022} \equiv 2^3 + (-1)^{2022} \equiv 0 + 1 \equiv 1 \pmod{4}.$$

Since $k \equiv 1 \pmod{4}$, we have $3^k \equiv 3^1 \pmod{10}$. Thus, $k^3 + 3^k \equiv 3 + 3 = \boxed{6} \pmod{10}$. □

Solution 2

a Note that $100 \equiv -1 \pmod{101}$, so then $10000 \equiv (-1)^2 \equiv 1 \pmod{101}$. Thus we can break the number into 2 digit numbers, and their alternating sum will be equivalent to the original number modulo 101:

$$\overline{2a191776b9} \equiv \overline{2a}(100^4) + 19(100^3) + 17(100^2) + 76(100) + \overline{b9} \equiv (20+a) - 19 + 17 - 76 + (10b+9) \pmod{101}.$$

This simplifies to $10b + a \equiv 49 \pmod{101}$. Then, clearly $\boxed{(a, b) = (9, 4)}$. □

Solution 3

To do so, we must compute the number but remove all factors of 10 (i.e. $2 \cdot 5$), and take the resulting number modulo 100. After doing this (removing 4 factors of each 2 and 5, the number is

$$6 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 2 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 3 \cdot 16 \cdot 17 \cdot 18 \cdot 19.$$

This can then be computed as $\boxed{64} \pmod{100}$. □

Solution 4

Since we are working in modulo 75, we have $66 \equiv -9 \pmod{75}$. In particular, we can write the given equivalence as

$$n^2 \equiv 6n - 9 \pmod{75} \implies n^2 - 6n + 9 \equiv 0 \pmod{75}.$$

Thus, $75 \mid (n-3)^2$. A common mistake here might be to conclude that $n \equiv 3 \pmod{75}$; however, this is false. Noting that $75 = 3^2 \cdot 5$, we have $3 \mid (n-3)^2 \implies 3 \mid n-3$, and $5^2 \mid (n-3)^2 \implies 5 \mid (n-3)$ (now, these implications are correct since 3 and 5 are primes). So, this condition is equivalent to $15 \mid n-3$. Then there are $\boxed{34}$ solutions: $15(0)+3, \ldots, 15(33)+3$. □

Solution 5

We will analyze this equation modulo 10. The key fact is that for $n \geq 5$, $n! \equiv 0 \pmod{10}$. Thus the right side of the equation will be the same modulo 10 for all $n \geq 5$; it will in fact be equivalent to $1+2+6+24 \equiv 3 \pmod{10}$. But no perfect square can be $3 \pmod{10}$, so only $n \leq 4$ can give solutions.

Upon testing each of these n, we find that $\boxed{(m,n) = (1,1),(3,3)}$ are the only solutions. □

Solution 6

Consider the x_n modulo 12. We have $x_0 = 2018 \equiv 2 \pmod{12}$; under the first operation, clearly $x_n \equiv x_{n-1} \pmod{12}$; under the second operation, if $x_{n-1} \equiv 2 \pmod{12}$, then we get $x_n \equiv 2 \pmod{12}$ as well. So, this implies all x_n must be equivalent to 2 modulo 12. From testing answer choices, we see that (D) satisfies this constraint.

However, we must also check that any number which is $2 \pmod{12}$ could be achieved as some x_n. But this is true because we can use the $x_n = 9x_{n-1} - 4$ operation to create arbitrarily large numbers which are $2 \pmod{12}$, and then keep subtracting 12 from them (the first operation) as needed. So, this confirms that the answer is $\boxed{\textbf{(D)}}$. □

Solution 7

Note that we can rewrite the equation as

$$18a - a + 6b = 18 - 5 \implies 18a + 6b - 18 = a - 5.$$

The left side is divisible by 6, so the right side must be as well. Thus, $6 \mid a - 5 \implies a = 6k + 5$ for some integer k. (Another way to see this is to just take the original equation modulo 6; simplifying will give this same result of $a \equiv 5 \pmod{6}$). Plugging this back into the original equation gives

$$17(6k+5) + 6b = 13 \implies b = -12 - 17k.$$

Thus, $a - b = (6k+5) - (-12-17k) = 23k + 17$, where k is any integer. Then clearly the smallest possible positive value for $23k + 17$ is $\boxed{17}$, when $k = 0$. □

Solution 8

Note that the number $\overline{abc}$ is equal to $100a + 10c + b$. Similarly expressing the other permutations of this number and summing, we see that the six numbers sum to $222(a+b+c)$ (because each a, b, c is in each of the 3 digit positions twice). That is, we have $\overline{abc} + 3194 = 222(a + b + c)$.

The nearest multiples of 222 greater than 3194 are $3330, 3552, 3774, 3996$. Since $\overline{abc}$ is a 3 digit number, one of these must give the answer. Testing them, we find that $\overline{abc} + 3194 = 3552$ gives $\overline{abc} = 358$, which also satisfies $3552 = 222(a + b + c)$. Thus the answer is $\boxed{358}$. ☐

Solution 9

By the given condition, note that $17 \mid 1 + 4a + 5b + 20ab$. So, $17 \mid (1 + 4a)(1 + 5b)$. Since 17 is prime, this means either $17 \mid 1 + 4a$ or $17 \mid 1 + 5b$ (or both). Note that since a, b range from 1 to 17, there will only be one value of a satisfying $4a \equiv -1 \pmod{17}$ and similar for b. For this value of a, there are 17 choices of b which will then satisfy the condition in the statement, and similar for the b satisfying $5b \equiv -1 \pmod{17}$. One of these pairs overlaps, so the answer is $17 + 17 - 1 = \boxed{33}$. ☐

Solution 10

Note that p divides the difference of those two expressions as well, i.e. $p \mid 2n + 8$. So, $2n \equiv -8 \pmod{p}$. For $p > 2$, we have $\gcd(p, 2) = 1$ so we can simplify to $n \equiv -4 \pmod{p}$.

Now, note that since $n^2 \equiv -1 \pmod{p}$ from the first condition, we have $(-4)^2 \equiv -1 \pmod{p} \implies 17 \equiv 0 \pmod{p}$. Thus, $p \mid 17$, so $p = 17$ is the solution in this case. Indeed, $p = 17$ and $n = 13$ satisfy the given conditions. The only other possible value of p is 2, but we want the largest, so the answer is $\boxed{17}$. ☐

Solution 11

Note that $100 \equiv -1 \pmod{101}$. So, if we write the six digit number as $\overline{abcdef}$, we can consider the digits in groups of two: $\overline{ab}(1000) + \overline{cd}(100) + \overline{ef}$. Clearly $100^2 \equiv 1 \pmod{101}$, so then this is equivalent to

$$(10a+b)(1000)+(10c+d)(100)+(10e+f) \equiv (10a+b)(1)+(10c+d)(-1)+(10e+f) \equiv 10(a-c+e)+(b-d+f) \pmod{101}.$$

We want this to be $0 \pmod{101}$. Now note that since each digit is between 1 and 5, each of the terms $a-c+e$ and $b - d + f$ ranges from -3 to 9. In particular, this implies $-101 < 10(a - c + e) + (b - d + f) < 101$. Thus it must equal 0. So $10(a - c + e) = -(b - d + f)$, but the magnitude of the right side is also less than 10, so both of these must in fact be 0.

So $b + f = d$. It is easy to see that there is 1 solution if $d = 2$, 2 if $d = 3$, and so on, for a total of $1 + 2 + 3 + 4 = 10$ solutions. There are then also 10 solutions to $a + e = c$, so the answer is $10 \cdot 10 = \boxed{100}$. ☐

Solution 12

Proof left for other weeks.

Solution 13

Claim (Fermat's Little Theorem): For prime p and integer a, $a^p \equiv a \pmod{p}$.
Proof. For $a \equiv 0 \pmod{p}$, it is obvious. Now, consider a with $\gcd(a, p) = 1$. Consider the set of the $p - 1$

numbers

$$\{a, 2a, \ldots, (p-1)a\}.$$

If some $ia \equiv ja \pmod{p}$ for $i \neq j$, then we have $a(i-j) \equiv 0 \pmod{p}$. But since i, j are distinct and from 1 to $p-1$, clearly $i - j \not\equiv 0 \pmod{p}$. So this would contradict the fact that $\gcd(a, p) = 1$. Thus, the above set consists of $p-1$ distinct numbers modulo p. These must in fact be $1, 2, \ldots, p-1$ in some order. So,

$$(a)(2a)(\cdots)((p-1)a) \equiv (1)(2)(\cdots)(p-1) \pmod{p} \implies a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}.$$

Since clearly $\gcd(p, (p-1)!) = 1$, we can then conclude $a^{p-1} \equiv 1 \pmod{p}$, as desired. □