PEANO AXIOMS: WHAT IS A NUMBER?

OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Induction

Induction is a very powerful tool for proving theorems. We will need it a lot in the development of addition and multiplication. Suppose that we have an infinite list of related mathematical statements S_n where n are natural numbers $1, 2, 3, ...$ The first statement is called the base case. Suppose that S_1 is true. If we establish the inductive step by proving that S_n implies S_{n+1}, then we prove the validity of the statements S_n for any and all natural n. Indeed, $S_1 \implies S_2$, $S_2 \implies S_3$, and so forth.

An example of mathematical induction is the domino effect. Imagine that we have an infinite set of dominoes lined up at equal distances along a straight line. Imagine further that the distance between the dominoes is short enough for a falling domino to force the fall of the next one.

Problem 1. Warm up. Prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

using induction.

2. What is a number?

There's a good chance you know *intuitively* what a real number is, but what if an alien lands on earth and doesn't know any math? Can you explain to them what a real number is? We aren't going to be able to answer this question, but we will get close. (If you are interested, feel free to ask your lead instructor(s) after class about the construction of the real numbers). We will be investigating how we can rigorously define the natural numbers, the integers, and finally the rational numbers using the Peano Axioms, named after Giuseppe Peano. Peano founded a large portion of mathematical logic and set theory that is integral to the foundations of modern mathematics.

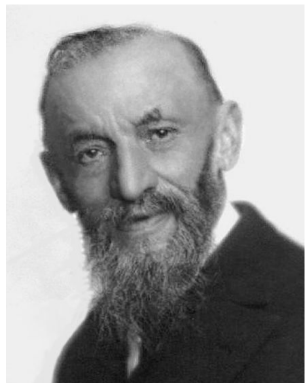

Figure 1. Giuseppe Peano

Problem 2. At your table, discuss with your fellow students what exactly a real number is. We are trying to be *constructive* in our definition of different types of numbers, so you can't define a real number in terms of what you can do with it. So the answer "a real number is a number you can add, subtract, etc." is not correct.

3. Peano Axioms

We are going to be constructing a "new" system of numbers, the Peano numbers. These will end up being the same as the numbers we already know, but we are going to be constructing them rather than using facts about them we already know. So, we need some Peano natural number to start with.

1. 0 is a Peano natural number.

0 is the Peano natural number we start with. For now, we don't know what exactly 0 is, just that it is a Peano natural number. To build the rest of the Peano natural numbers from 0, we have a successor function S that represents adding one.

2. For every Peano natural number n, $S(n)$ is a Peano natural number.
3. For all Peano natural numbers m and n, $m = n$ if and only if $S(m) = S(n)$.
4. For every Peano natural number n, $S(n) = 0$ is false.

The Peano natural numbers will be the set of 0, $S(0)$, $S(S(0))$, and so on. This is not the only way of constructing the naturals, but it is a convenient one and the one we will be using. It is reasonable to think we are done, but there is one nuance we have overlooked.

Problem 3. Define a set $S = \{0, S(0), S(S(0)), \dots\} \cup \{a, b\}$ where $S(a) = b$ and $S(b) = a$. Which of the Axioms 1-4 does this set S satisfy? Is this an issue?

As a result, we introduce the final axiom.

5. (Axiom of Induction) Let K be a set such that $0 \in K$ and $n \in K$ implies $S(n) \in K$ for every Peano natural number n. Then K contains every Peano natural number.

Problem 4. Explain how the Axiom of Induction deals with the issues outlined in Problem 3.

You might notice some similarities between the Axiom of Induction and the proof technique we have discussed called Induction. In fact, Induction works only because we have the Axiom of Induction! Say we have a statement P_n that is true depending on n. For example, we might have P_n be the statement $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Then, we take a set J of all Peano natural numbers n where P_n is true. That is, $J \subseteq \mathbb{N}$ and $n \in J$ if and only if P_n is true. We start by proving P_0 is true as our base case, so $0 \in J$. Then, as our Inductive Step, we prove that if P_n is true, then so is P_{n+1}. (For now, let us assume that $S(n) = n + 1$, we will prove it later). Then, this is the same as saying that if $n \in J$, then so is $S(n)$. But then by the Axiom of Induction, $J \supseteq \mathbb{N}$. We already know $J \subseteq \mathbb{N}$, so $J = \mathbb{N}$. Thus, P_n is true for every n, and we have proven what we wish to prove using the Axiom of Induction.

4. Representation of Peano Natural Numbers

We know that 0 is a Peano natural number, but what is 0 actually? The solution to this problem comes in the form of Von Neumann ordinals, named after John von Neumann. Von Neumann is known for his numerous contributions to mathematics and physics (he even worked on the Manhattan Project to develop the atomic bomb), but in particular, we are interested in his rigorous development of set theory into the modern field it is today.

Figure 2. John von Neumann

We want to define every Peano natural number, starting from 0. All we have at this point is sets, so we will construct the Peano natural numbers in their entirety just from sets. We start by defining $0 = \emptyset$ (the empty set). Then, we define $S(x) = x \cup \{x\}$. Thus, $1 = S(0) = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}$. Similarly, $2 = S(1) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$. Then, we let $\mathbb{N} = \{0, S(0), S(S(0)), \dots\}$.

Problem 5. What is 3 as a Von Neumann ordinal?

We won't be discussing inequalities so take this as an interesting example rather than a new topic.

Problem 6. If a and b are both Peano natural numbers, and $a \leq b$ (using our usual definition, we haven't precisely defined this), what can we say about the Von Neumann ordinals for a and b?

Problem 7. Prove that this definition of $\mathbb{N}$ satisfies the Peano Axioms.

5. Peano Arithmetic

5.1. Peano Addition.

Problem 8. Peano Addition is a function that takes two Peano natural numbers as input and spits out one Peano natural number as output. Come up with a definition for Peano addition of Peano natural numbers using only the Peano axioms. (Hint: Use a recursive formula, using $S(n)$.)

Definition 1. To make sure we are all on the same page, define *Peano addition* of Peano natural numbers recursively as follows:

(i) $m + 0 = m$ for all Peano natural numbers m,
(ii) $m + S(n) = S(m + n)$ for m and n Peano natural numbers.

Problem 9. In the following examples, write explicit formulas in terms of a Peano natural number m and S.

(a) $m + 1$
(b) $m + 2$

Problem 10. Corollary: prove that $1 + 1 = 2$.

Write solutions to the following problems justifying each step with axioms, definitions, and previously proven results. It will be helpful to make frequent use of the Axiom of Induction. Remember that while these problems may seem very obvious, we are using Peano addition and not the addition you already know, so we don't yet know these statements to be true!

Problem 11. For the base case of the recursive definition of Peano addition, we took $m + 0 = m$. Prove that $0 + n = n$ for any Peano natural number n.

Problem 12. For the recursive part of the definition of Peano addition, we took $m + S(n) = S(m + n)$. Prove that $S(m) + n = S(m + n)$ for any Peano natural number m.

Problem 13. Prove that Peano addition of Peano natural numbers is associative. In other words,

$$(\ell + m) + n = \ell + (m + n)$$

for any Peano natural numbers ℓ, m, and n.

Problem 14. Prove the Peano addition of Peano natural numbers is commutative or

$$m + n = n + m$$

for any Peano natural numbers m and n.

You may find it interesting to know that the algebraic structure $(\mathbb{N}, +)$ where the sum of any two Peano natural numbers is still a Peano natural number is called a *monoid*. This is far beyond our scope, however.

5.2. **Peano Multiplication.**

Problem 15. Define Peano multiplication of Peano natural numbers using only Peano axioms and Peano addition of Peano natural numbers. Your definition will probably look very similar to your recursive definition of Peano addition.

Problem 16. Prove that $S(0)$ is the Peano multiplicative identity. In other words, show

(a) $m \cdot S(0) = m$,
(b) $S(0) \cdot n = n$.

Problem 17. Prove that Peano multiplication distributes over Peano addition or

$$\ell \cdot (m + n) = \ell \cdot m + \ell \cdot n$$

for all Peano natural numbers ℓ, m, and n.

If you are looking for more formalisation of our Peano number system, prove that Peano multiplication is associative and commutative (this is optional).

6. Peano Integers

As you have probably noticed, all the Peano natural numbers are nonnegative. This means that we don't have "Peano additive inverses": we know from our previous knowledge of numbers that $a+(-a)=0$ (which would show that $-a$ is the additive inverse of a), but we haven't defined what $-a$ is! All we know are nonnegative numbers!

Definition 2. The *Peano additive inverse* of a Peano natural number a is denoted by $-a$ and is any number with the following properties:

1. $a+(-a)=0$
2. $a+(-b)=c+(-d)$ if and only if $a+d=b+c$
3. $(-a)*b=a*(-b)=-(a*b)$

Problem 18. (*Optional*) Prove that the Peano additive inverse is unique.

Problem 19. Show that $-(-a)=a$ for any Peano natural a.

Problem 20. Show that $-0=0$.

These two properties are enough to define Peano subtraction: $a-b=a+(-b)$. This is unique because $-b$ is the unique Peano additive inverse of b.

Problem 21. Prove that $a-0=a$ for any Peano natural a.

Problem 22. Prove that $3-2=1$.

Problem 23. Prove that $2-3=-1$.

Problem 24. Prove that $(-a)*(-b)=a*b$, for any Peano integers a,b.

Optional bonus information: You may notice this is not a construction of Peano integers. We never said what "$-a$" actually is. The construction of the Peano integers (and later the Peano rationals) are a little bit complicated, so they will be given for completeness but there won't be any problems about them. Define a *Peano integer* as a pair of Peano naturals, $x=(n,m)$. Then, we define $x=(n,m)=y=(o,p)$ if and only if $n+p=o+m$. We define $-x=(m,n)$ and $x+y=(n+o,m+p)$. This strange notation corresponds to notation you may have seen before, $(n,m)=n-m$, but without using any ideas of subtraction. For convenience, as $(1,0)=(2,1)=\dots$, we denote any of these equivalent Peano integers with the symbol $1=(1,0)$, and we denote 1's Peano additive inverse as $-1=(0,1)$. Then we can write the set of the Peano integers with the symbol $\mathbb{Z}=\{0,1,-1,2,-2,\dots\}$. You may find it interesting to know that the algebraic structure $(\mathbb{Z},+,*)$ where Peano addition is invertible but Peano multiplication isn't invertible is called a *ring*. This is far beyond our scope, however.

7. Peano Rational Numbers

Now, we have Peano additive inverses, but what about Peano multiplicative inverses?

Definition 3. The *Peano multiplicative inverse* of a Peano integer a is denoted by $\overline{a}$ (we are not using the notation $\frac{1}{a}$ because we have not yet defined Peano division) and is any number with the following properties:

1. $a * \overline{a} = 1$
2. $a * \overline{b} = c * \overline{d}$ if and only if $a * d = b * c$ (you may know this as "cross multiplication")
3. $-\overline{a} = \overline{-a}$

Problem 25. (*Optional*) Prove that the Peano multiplicative inverse is unique.

Problem 26. Show that $\overline{\overline{a}} = a$ for any Peano integer a.

Problem 27. Show that $\overline{1} = 1$.

These two properties are enough to define Peano division: $\frac{a}{b} = a * \overline{b}$.

Problem 28. Show that $\frac{a}{1} = a$ for any Peano integer a.

Problem 29. Show that $\frac{1}{a} = \overline{a}$ for any Peano integer a.

Problem 30. Prove that $\frac{6}{3} = 2$.

Problem 31. Prove that $\frac{3}{6} = \frac{1}{2}$.

Problem 32. Show that $\frac{-a}{-b} = \frac{a}{b}$ for any Peano integers a, b.

Problem 33. Prove that $\frac{a*b}{a} = b$, for any Peano integers a, b.

Optional bonus information: Using a similar process to the Peano integers, we can construct Peano rationals as a pair of Peano integers, $q = (a, b)$, corresponding to the more familiar notation $q = \frac{a}{b}$ but without using any concepts of division. Then if $r = (c, d)$, we define $q = r$ if and only if $ad = bc$, $q + r = (ad + bc, bd)$, $\overline{q} = (b, a)$, and $qr = (ac, bd)$. We denote the set of the Peano rationals with the symbol $\mathbb{Q}$. You may find it interesting to know that the algebraic structure $(\mathbb{Q}, +, *)$ where Peano addition and Peano multiplication are invertible (except that there is no Peano multiplicative inverse of 0) is called a *field*.