3D Geometry with Quaternions

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1 Quaternions

1.1 Complex Numbers and Geometry

To start this worksheet, let's contrast two ways of understanding the 2D plane: vectors and complex numbers. In the 2D case, a *vector* is just an ordered pair (x, y) denoting an x and a y coordinate. Vectors are like numbers in so far as you can add them, subtract them, and multiply them by real numbers, but multiplying vectors turns out not to mean much geometrically.

Definition 1. The *magnitude* of a point on the plane (or in any Euclidean space) is the distance from that point to the origin (in the usual Euclidean metric).

Problem 1. Give a formula for the magnitude of the point (a, b).

Describe how you can use complex number operations, such as addition, subtraction, multiplication, division, and/or conjugation, to find the magnitude of the point (a, b), thought of as the complex number $a + bi$.

Problem 2. What happens to the magnitude of complex numbers when they are multiplied?

Problem 3. Describe how to scale up a point $a+bi$ by a real-number factor r using complex number operations.

Problem 4. Describe how to use complex number operations to rotate the point $a + bi$ counterclockwise around the origin by an angle of θ, measured in radians.

(Hint: Start by finding an expression for the coordinates of the point you get by rotating $1 + 0i$ by an angle of θ. Then try doing the same to $0 + 1i$ or other easy vectors.)

Problem 5. One observation that will be useful before the next section: the set $\{1, -1, i, -i\}$ is closed under multiplication. Draw a multiplication table for these numbers.

This multiplication should satisfy a sort of "Sudoku principle" - every row and column should contain each number exactly once. Explain why.

1.2 Introducing j and k

The system of using complex numbers, which we can multiply, to represent points in the plane $\mathbb{R}^2$ is phenomenally useful when it comes to describing geometric actions like rotation. But can we do anything similar for 3D space?

Problem 6. The first thing we might think to do is to take the complex numbers and throw in another square root of -1, which we'll call j. Explain why this won't work (at least, why the set $\{\pm 1, \pm i, \pm j\}$ can't be closed under multiplication, with a multiplication table that has each of the 6 numbers appear exactly once in each row and column).

Problem 7. Miraculously, this problem goes away when we instead try to add *yet another* square root of -1, which we'll call k, although we end up with new problems. Specifically, we have to lose *commutativity* - the property where $xy = yx$, although we will make sure that all real numbers, and in particular ± 1, still commute with everything (so if x is real, $xy = yx$).

Show that ij must either be k or $-k$. From this point on, we'll set $ij = k$. Find the multiplication table for $\{\pm 1, \pm i, \pm j, \pm k\}$. Because it won't be commutative, let the entry in row x and column y represent xy.

In particular, take note of what ij, jk, and ki are.

Convince yourself that $ij = -k$ would have worked also.

We can now introduce a number system known as the *quaternions*, because it's fundamentally 4-dimensional.

Definition 2.
- Let $\mathbb{H}$ be the set of all $a + bi + cj + dk$ where a, b, c, d are real numbers, and i, j, k multiply as described in Problem 7.
- Elements of $\mathbb{H}$ are called *quaternions*.
- The *real part* of the quaternion $q = a + bi + cj + dk$ is $\operatorname{Re} q = a$.
- The *imaginary part* or *vector part* of the quaternion $q = a + bi + cj + dk$ is $\operatorname{Im} q = bi + cj + dk$.
- If v is a quaternion with $\operatorname{Re} v = 0$, we call v a *vector quaternion*. These are what we will use to study points in 3D space.

However, we still have some quaternion basics to do before we focus on 3D.

Problem 8. As with complex numbers, define the magnitude of the quaternion $a + bi + cj + dk$ to be $|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Find an appropriate definition of the conjugate of a quaternion $q = a + bi + cj + dk$, and show that $|q| = \sqrt{q\bar{q}}$, and that if $q, r \in \mathbb{H}$ are quaternions, then $|qr| = |q||r|$.

Problem 9. Given a nonzero quaternion $q = a + bi + cj + dk$, find a formula for the quaternion q^{-1} such that $q^{-1}q = 1$. Show that $qq^{-1} = 1$ also.

Optional: Show that q^{-1} is unique - that is, there is only one quaternion satisfying $q^{-1}q = 1$ or $q^{-1}q = 1$.

2 Quaternion Products in 3D Geometry

To do 3D geometry with quaternions, we will start by looking at what happens when we take two vector quaternions, $v = ai + bj + ck$ and $w = di + ej + fk$, multiply them, and split the answer into its real and imaginary part.

Definition 3.
- Define $v \cdot w = -\operatorname{Re}(vw)$. This real number is called the *dot product*.
- Define $v \times w = \operatorname{Im}(vw)$. This vector quaternion is called the *cross product*.

Problem 10. Calculate the following dot and cross products:

- $(3i + 2j) \cdot (i - j)$
- $(3i + 2j) \times (i - j)$

2.1 The Dot Product and Angles

Problem 11. Find a formula for $v \cdot w$ in terms of the coordinates of v and w.

Problem 12. Is the dot product commutative?

Problem 13. Explain how the dot product relates to the magnitude of a vector.

Problem 14. Now we will find out what the dot product means *geometrically*.
Let a, b be two vector quaternions, and consider the following triangle:

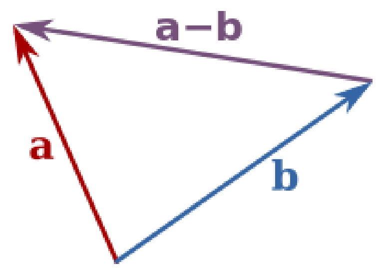

Figure 1: The triangle with corners $0, a, b$.

- Use the dot product to find expressions for the lengths of all three sides of this triangle.
- Use the law of cosines to find a geometric expression for $a \cdot b$ which makes no reference to the specific coordinates of a and b.

Problem 15. Two vectors are called *orthogonal* when $v \cdot w = 0$. Explain geometrically what this means, and find a synonym for the word "orthogonal". (After you solve this, you can ask your instructor for the difference between these two words.)

Definition 4. If P is a 2D plane in 3D space, a *normal vector* to P is a vector orthogonal to all differences $v - w$ where v, w are in P.

Problem 16. If P is a plane with normal vector N, given as a vector quaternion N, and q_0 is a point on P, then find an equation of the form $ax + by + cz = d$ for P in terms of the coordinates of N and q_0.

Problem 17. Given a plane defined by an equation as described in the last problem, and a point in space, show that $\frac{N \cdot (q - q_0)}{|N|}$ is the shortest distance from that point to the plane.

Problem 18 (Optional)**.** Given the equation of a plane and a point in space, explain how to project that point onto that plane perpendicularly. That is, imagine that there was a light shining from very, very far away from the point and the plane, so that the point cast a shadow which hits the plane at a right angle. Where does that shadow lie?

2.2 Cross Product

Problem 19. Using Problem 14, find a formula for the magnitude $|v \times w|$, without referencing explicit coordinates.

Problem 20. If v, w are vector quaternions, with $\operatorname{Re} v = \operatorname{Re} w = 0$, then how does vw differ from wv?

Problem 21. Is the cross product commutative? If not, what is the difference between $v \times w$ and $w \times v$?

Problem 22. Find algebraic expressions (using only addition, subtraction, multiplication, and division, but not Re or Im) for $v \cdot w$ and $v \times w$.

Problem 23. Compute $v \times v$.

Problem 24. Compute $v \cdot (v \times w)$.

Problem 25. In Problem 19, you determined the magnitude of $v \times w$. Describe the direction of $v \times w$.

(Hint: You may need to describe the standard way of drawing the x, y, z axes, or as we might call them, the quaternions i, j, k. For that, consider the following Swiss bank note:)

Figure 2: The right-hand rule, per wikimedia

Problem 26. Say that a plane contains points u, v, w.

- Find a normal vector to this plane.
- Find an equation for this plane using that normal vector (and dot and cross products).

Problem 27. Find a formula for the area of the parallelogram with corners $0, v, w, v + w$, and relate it to the dot and/or cross product. Then if u is also a vector quaternion, find a formula for the volume of the paralellepiped (basically a 3D parallelogram, or a skewed cube) with corners $0, u, v, w, u + v, u + w, v + w, u + v + w$.

3 Quaternions and 3D Rotation

In this section, we will see how to use quaternions to perform 3D rotations about the origin, in analogy to the way we used complex numbers to perform 2D rotations. It won't be quite as simple, but that's because 3D rotations aren't as simple as 2D! Our points in 3D space will be represented by vector quaternions (quaternions with $\operatorname{Re} q = 0$, of the form $ai + bj + ck$), and we will try to understand how to rotate these with quaternion multiplication. This procedure is often used to model rotations in computer graphics.

To better visualize the material in this section, please check out the fantastic videos and interactive visualizations by Grant Sanderson (3Blue1Brown) and Ben Eater here after class. In the meantime, the following theorem will be useful for visualizing and talking about rotations:

Theorem 3.1 (Euler's Rotation Theorem)**.** *Every rotation of 3D space around the origin fixes an axis, known as the* Euler axis*, which is unique unless the rotation is the identity. The rotation is uniquely determined by the axis and the angle of rotation around the Euler axis.*

In order to define the angle of rotation around the Euler axis without confusion about sign conventions, we will use a right-hand rule.

Definition 5. Define $R_{v,\theta}$ to be the rotation we get by pointing our right-hand thumb in the direction of v, and rotating around the axis through v for θ radians in the direction given in this picture:

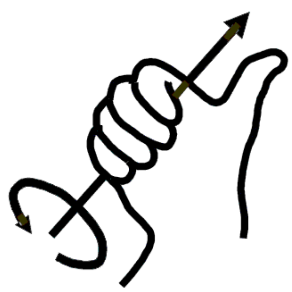

Figure 3: The right-hand rule for rotations, from wikimedia.

Problem 28. Explain why 3D rotation can't simply be described by multiplying on the left (or the right) by a quaternion.

Problem 29. In order to understand rotations, we will first have to do a few calculations: If $q = a + bi + cj + dk$, evaluate iqi, jqj, and kqk. Explain what these operations do geometrically if $a = 0$ and q is a vector.

Problem 30 (Optional)**.** Use Problem 29 to find an expression for the real and imaginary parts of a quaternion, using only addition, subtraction, multiplication, and division. (This cannot be done for the complex numbers!)

Problem 31. If v is a vector quaternion, explain how to use quaternion algebra to rotate v 180° about the i-, j-, or k-axis.

Problem 32. We've now seen that multiplying by quaternions on *both* sides can rotate vectors. Let q be a quaternion. Show that if v is a vector quaternion, then qvq^{-1} is as well.

Problem 33. If the function $R(v) = qvq^{-1}$ is a rotation, what is the fixed axis of this rotation?

Problem 34. Find an expression for a rotation of θ radians (in the direction that moves j towards k) around the i-axis.

Problem 35 (Challenge)**.** Prove that if the map sending v to qvq' is a rotation, then $q' = q^{-1}$.

Problem 36. Give a formula for a rotation of angle θ about the axis of a vector v, assuming that $|v| = 1$. (It's ok for this to be an educated guess based on earlier examples.)

Problem 37. Call the function from your previous answer $R_{v,\theta}(w)$, so that $R_{v,\theta}(w)$ should ideally be the rotation of w around the axis of v by an angle of θ. As a check that this actually works, see what $R_{v,\theta}$ does to a vector orthogonal to v:

If w is orthogonal to v, check that $v \cdot R_{v,\theta}(w) = 0$, and that $w \cdot R_{v,\theta}(w)$ is what it should be.

Finally, check that you are rotating in the correct direction, by making sure that if you rotate by $90°$, or $\frac{\pi}{2}$ radians, that $R_{v,\theta}(w) = v \times w$, and that this is what the right-hand rules should give you.