ORMC Olympiad Group
Winter: Week 2
Modular Arithmetic II and Fermat's Thm

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Problems

1. **Wilson's Thm:** Let p be a prime number. Then

$$(p-1)! \equiv -1 \mod p$$

2. **(Euler's Phi Function)** For $n > 1$, $\phi(n)$ is define to be the number of integers among the set $\{1, 2, 3, \ldots, n-1\}$ that are relatively prime with n. For example, $\phi(5) = 4, \phi(6) = 2$

3. Prove *Euler's Theorem.*

   Euler's Thm Let $n > 1$ be any number, and a is an integer that is relatively prime with n. Then $a^{\phi(n)} \equiv 1 \mod n$

   Note that $n = p$ prime case is exactly **Fermat's Thm**

4. Find the number of nine digit integers in the form

$$*231*312*$$

   which are divisible by 37. Here $*$ represents a single digit, but all three can be different.

5. For any two digit integer $n = \overline{ab}$, define $S_n = 121314\ldots\overline{ab}$, so that all two digit numbers from 12 to $n = \overline{ab}$ are put together in order. For example, $S_{14} = 121314, S_{32} = 121314\ldots3132$. Find the smallest two digit number $n \geq 12$ so that $99|S_n$

6. Let p be a prime number, so that both of the numbers $p^2 + p + 1$ and $p + 10$ are also primes. Find p.

7. Prove that among n numbers there are $1 \leq k \leq n$ numbers which their sum is divisible by n.

8. Find the remainder when $2^{107} + 3^{107} + 6^{107}$ divided by 109.

9. (JBMO 2008 - shortlist) Is it possible to arrange the numbers $1^1, 2^2, 3^3, \ldots, 2008^{2008}$ one after the other, in such a way that the obtained number is a perfect square? (Explain your answer.)

10. Find all positive integers (n, m, k) such that $k \geq 2$ such that

$$1! + 2! + \cdots + n! = m^k$$

11. **(JBMO-2007)** Let p be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.

12. A number m in *mod* n is called perfect square if there exist an integer x such that $m \equiv x^2 \mod n$. For example, $0, 1, 4$ are perfect squares in *mod* 5, and $0, 1, 2, 4$ are perfect squares in *mod* 7. Find the total number of perfect squares in mod 25.

13. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 1, \ldots, a_{n-1} + n - 1$ is also a complete residue class?

14. For which n, there exist complete residue class $a_0, a_1, \ldots, a_{n-1}$ in $\mod n$ so that $a_0, a_1 + 3, a_2 + 6, \ldots, a_{n-1} + 3(n-1)$ is also a complete residue class?

15. Let $p > 2$ be a prime number. Is there a complete residue class $\{a_1, \ldots, a_{p-1}\} = \{1, 2, 3, \ldots, p-1\}$ in $\mod p$ so that $a_1, 2a_2, 3a_3, \ldots, (p-1)a_{p-1}$ is also a complete residue class?
HINT: Consider the product and use Wilson's theorem