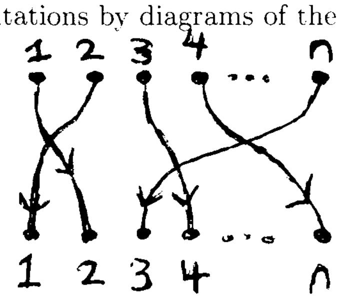
Braids and Groups

March 4, 2012

1 Permutations

Definition 1.1. A *permutation* is a function $f : \{1, 2, 3, \dots, n\} \to \{1, 2, 3, \dots, n\}$ with the property that for each integer $i \in \{1, 2, \dots, n\}$ there is a unique $j \in \{1, 2, \dots, n\}$ such that $f(j) = i$.

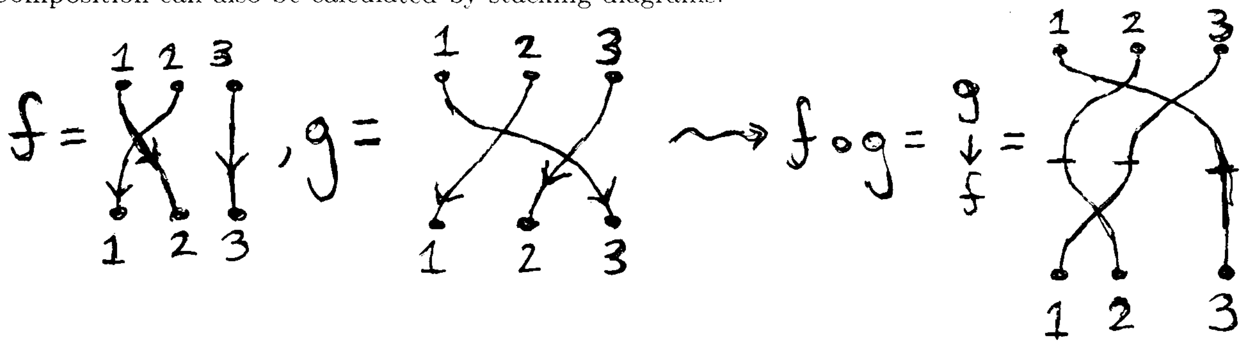
We can also represent permutations by diagrams of the form:



An arrow points from j to i if and only if $f(j) = i$.

Definition 1.2. We can combine two permutations f and g by taking the *composition* of the two functions, $f \circ g$. The composition $f \circ g$ is defined by $(f \circ g)(i) = f(g(i))$ for each $i \in \{1, 2, \dots, n\}$.

Composition can also be calculated by stacking diagrams:



2 Problems: Permutations

1. How many permutations of $\{1, 2, 3, \ldots, n\}$ are there?

2. Let $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ be defined by

$$f(1) = 4, \quad f(2) = 3, \quad f(3) = 5, \quad f(4) = 1, \quad f(5) = 2.$$

What is $f \circ f$? What about $f \circ f \circ f$? $f \circ f \circ f \circ f \circ f \circ f$?

3. Is it true that $f \circ g = g \circ f$ for any two permutations f and g of $\{1, 2, 3, \ldots, n\}$?

4. Prove that a composition of permutations is a permutation.

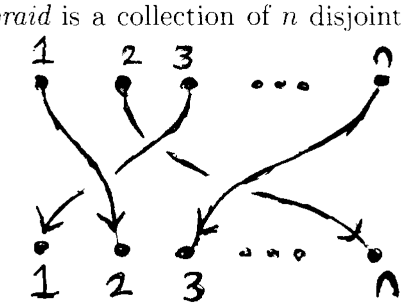
5. How are the symmetries of an equilateral triangle the same as the permutations of $\{1, 2, 3\}$? Is it true in general that the symmetries of a regular n-sided polygon correspond to the symmetries of $\{1, 2, 3, \ldots, n\}$?

6. Is it true that for any permutation f there is a permutation g such that $f \circ g$ satisfies $f \circ g(i) = i$ for all $i \in \{1, 2, \ldots, n\}$ (such a g is called an *inverse* for f)?

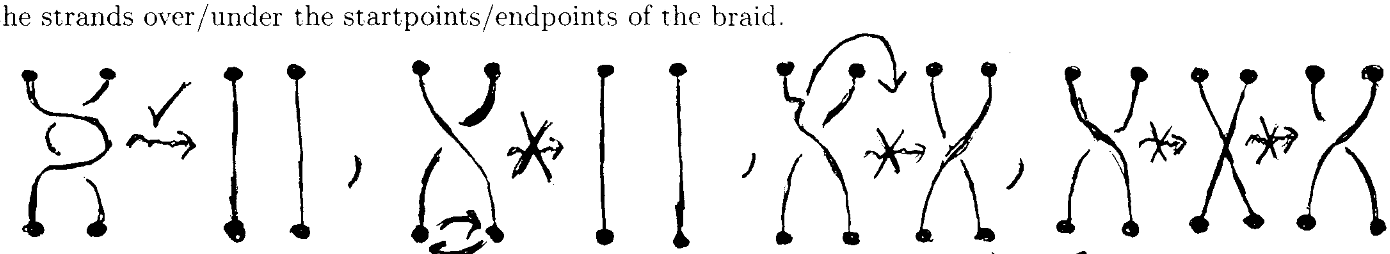
7. Prove that for each permutation f, there exists a number k such that composing f with itself k times gives the identity function (in other words, for each $i \in \{1, 2, \ldots, n\}$, $\underbrace{(f \circ f \circ \ldots \circ f)}_{k}(i) = i$).

3 Braids

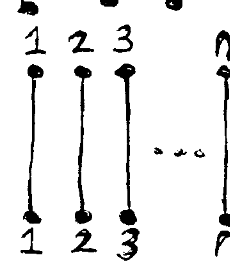
Definition 3.1. An *n-strand braid* is a collection of n disjoint strands in space, which are always pointing downward.



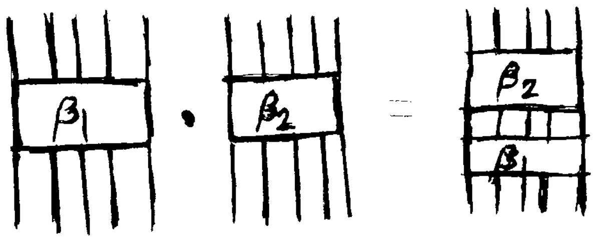
Definition 3.2. Two braids are *isotopic* if there is a way to deform one braid into another without 1) moving the startpoints/endpoints of the braid, 2) crossing strands, or 3) lifting the strands over/under the startpoints/endpoints of the braid.



Definition 3.3. An n-strand braid is *trivial* if it is isotopic to the braid



Definition 3.4. If β_1 and β_2 are n-strand braids, $\beta_1 \cdot \beta_2$ is the n-strand braid given by stacking β_2 on top of β_1.



Definition 3.5. An *inverse* of an n-strand braid β is an n-strand braid β^{-1} such that $\beta \cdot \beta^{-1}$ is trivial.

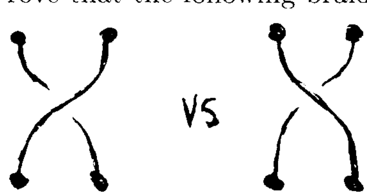
Theorem 3.6 (Fadell-Neuwirth). *If β is a non-trivial n-straid brand, then $\underbrace{\beta \cdot \beta \cdot \ldots \cdot \beta}_{k}$ is never trivial for any k.*

4 Problems: Braids

1. Find the inverse of the following braid.

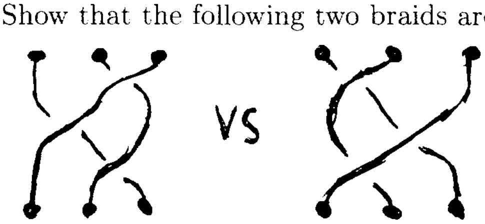
2. Prove that the following braid is not trivial.

3. Prove that the following braids are not isotopic.



4. Prove that any braid β has an inverse.

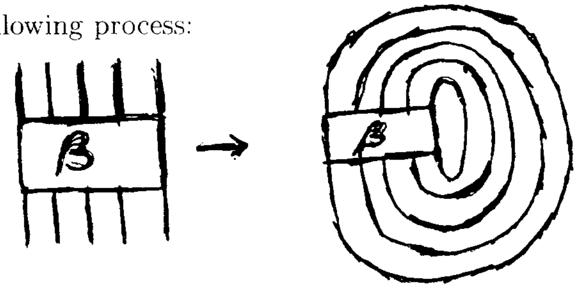
5. Show that the following two braids are isotopic.



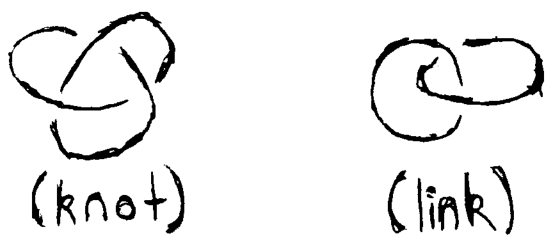
6. Prove that there are infinitely many different (non-isotopic) two-strand braids.

5 Knots

Definition 5.1. The *closure* of an n-strand braid is the collection of circles obtained by the following process:

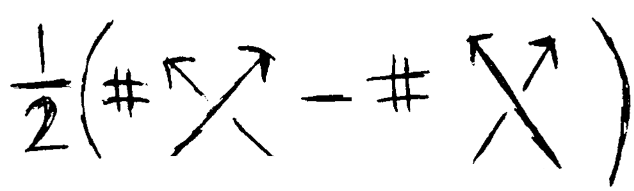


Definition 5.2. A *knot* is a circle in three-space that does not intersect itself. A *link* is a set of knots which do not intersect each other.



Definition 5.3. Two links are *isotopic* if one can be deformed into the other without self-intersections.

Definition 5.4. The *linking number* of two knots is defined by



In the formula, each crossing that is being counted has one strand from each knot.

6 Problems: Knots

1. Determine the conditions on a braid such that the closure will be a knot instead of a link. More generally how can we tell how many components the closure of the braid has?

2. Let β and γ be two n-strand braids. Prove that the closure of $\beta^{-1} \cdot \gamma \cdot \beta$ is isotopic to the closure of γ.

3. Construct a braid that closes up to a link which is two knots, each a non-trivial knot, but the two knots are not linked to each other.

4. Construct a link which consists of two knots that cannot be unlinked from each other, but still have linking number zero (you don't have to prove that they are actually linked...just draw it).