Graph Theory I - Properties of Trees

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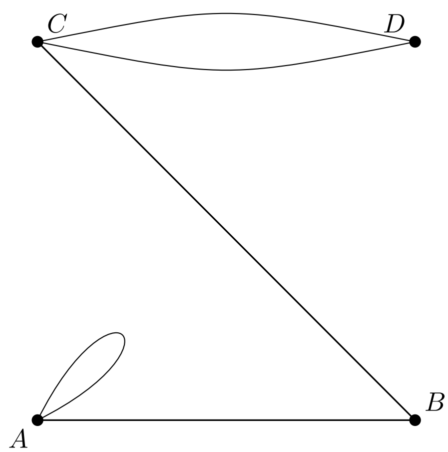
1 Graphs

Definition 1 *A* ***graph*** *G is a set $V(G)$ of points (called vertices) together with a set $E(G)$ of edges connecting the vertices.*

Though graphs are abstract objects, they are very naturally represented by diagrams, where we (usually) draw the vertices and edges in the plane. For instance, if we let G be the graph defined by $V(G) = \{A, B, C, D\}$ and $E(G)$ a set containing one edge connecting A and A, one connecting A and B, one connecting B and C, and two connecting C and D, we might represent it as follows.



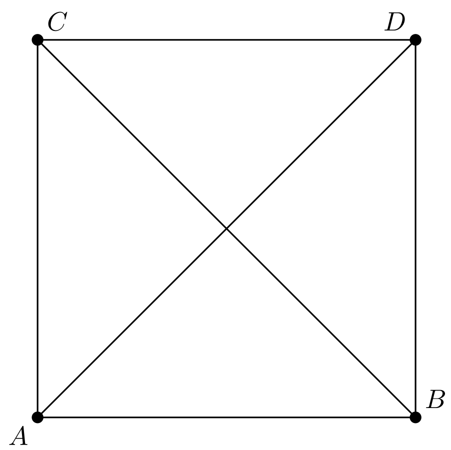
To eliminate a lot of bad behavior, we will also require some special properties of our graphs. As we'll see, almost all real-world applications of graphs do satisfy these properties, so it's not unreasonable to require them.

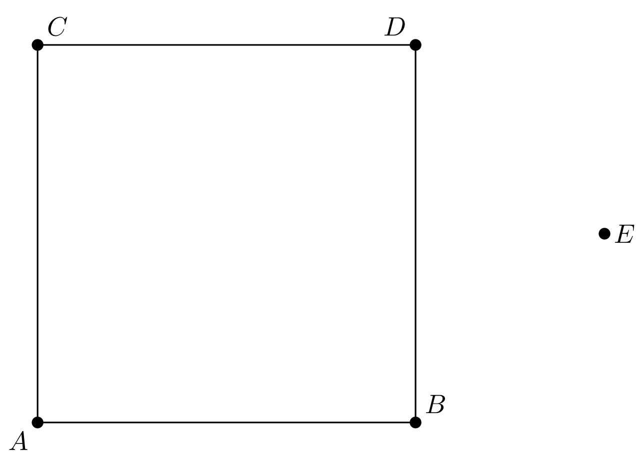
Definition 2

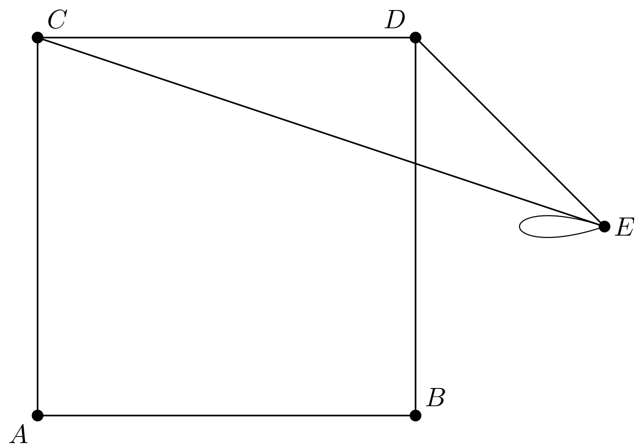
- *A graph is* ***connected*** *if every vertex is connected to some other vertex by an edge.*
- *A graph is* ***simple*** *if no vertex is connected to itself by an edge, and any two different vertices are connected by at most one edge.*

The graph above is *connected*, but is *not simple.*

Problem 1 *For each graph below, decide whether it is connected, and also decide whether it is simple.*







Problem 2 *For each of the following graphs, which is given to you by describing the sets of vertices and edges, determine whether it is connected, and also determine whether it is simple. Try to do this without drawing the graph if you can.*

- $V(G) = \{A, B, C, D\}$, $E(G)$ contains one edge connecting A and B, one connecting B and D, and one edge connecting A and D.

- $V(G) = \{A, B, C, D, E\}$, and $E(G)$ contains one edge connecting every pair of vertices.

- $V(G) = \{A, B, C, D, E\}$, and $E(G)$ contains one edge connecting every pair of *different* vertices.

As previously mentioned, we can represent many real-world situations by graphs. For instance, if we have a group of people, some of whom are friends with each other, we can represent this as a graph by letting our vertices be the people, and connecting two people with an edge if they are friends. Assume that everyone has at least one friend in this group.

Problem 3 *Is this graph connected? Is it simple?*

Problem 4 *For each example below, determine if the resulting graph is connected, and also determine whether it is simple.*

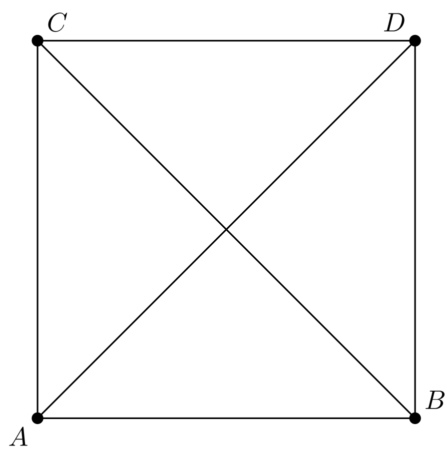
- A family tree, where people are related if one is the other's parent/child.
- A set of bus stations, where stations are related if there is a bus which goes directly from one to the other.
- A molecule, where atoms are related if they're bonded.
- A contact tracing network, where two people are related if they've been in close enough contact to spread an infectious disease.

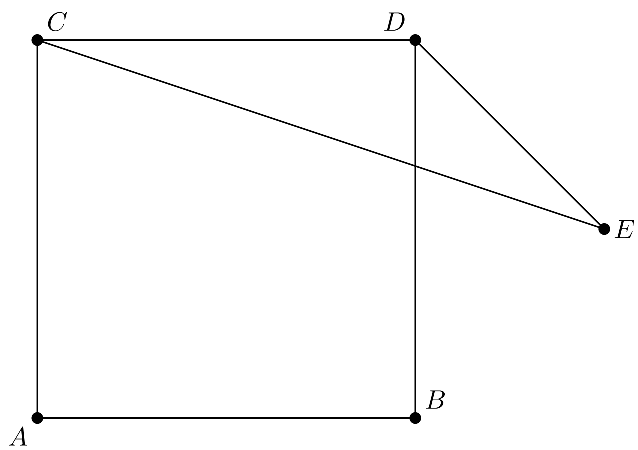
2 Degrees of Vertices

From now on, *all graphs are connected and simple.*

Definition 3 *The **degree** of a vertex $x \in V(G)$ is the number of other vertices connected to it.*

Problem 5 *For the following graphs, label the degree of each vertex.*





Problem 6 *Using the fact that G is simple, show that the degree of a vertex x is the same as the number of edges coming out of it.*

Problem 7 *Show that the sum of the degrees of every vertex of G is even.*

Problem 8 *Use the previous problem to show that in a group of seven people, not everyone can have exactly three friends.*

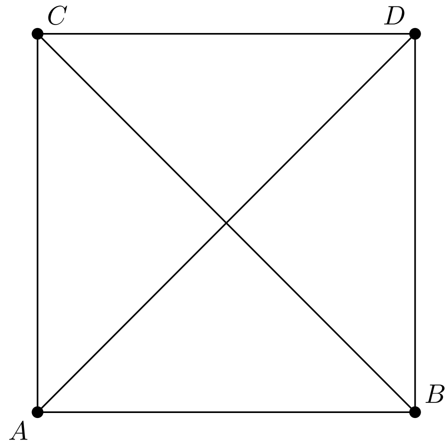
3 Trees

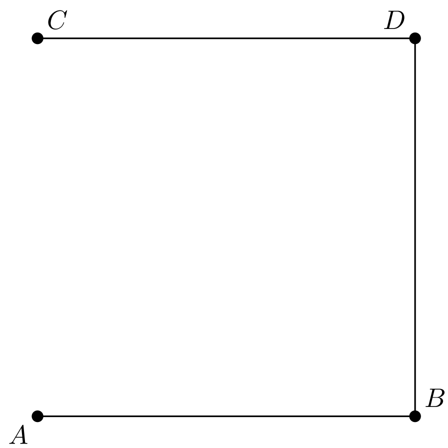
Definition 4 *Given a graph G,*

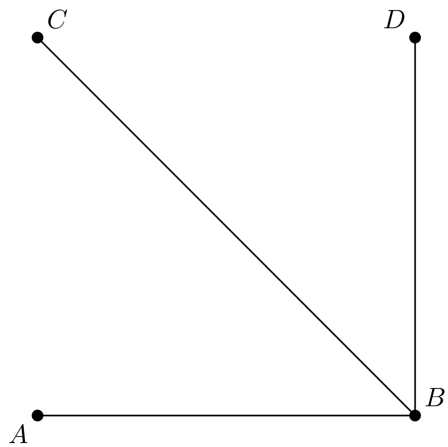
- *A **path** in G is a sequence of edges such that each edge begins where the previous edge ends and ends where the next edge begins.*
- *A **cycle** in G is a path starting and ending at the same vertex.*

*G is called a **tree** if it contains no cycles.*

Problem 9 *For each of the following graphs, determine whether or not it is a tree.*







Problem 10 *Show that every family tree is a tree.*

Problem 11 *Show that every tree with an edge has a vertex of degree* 1*. Such a vertex is called a* ***leaf*** *of the tree. (Hint: Consider the longest path in* G*, and take one of its endpoints* x*.* x *has to be connected to one other vertex because it's on this path, so show that it can't be connected to any other vertices.)*

Problem 12 *Let G be a tree, and let V be the number of its vertices, and E the number of its edges. Let us prove that*

$$E = V - 1$$

We will prove this by *induction on the number of vertices.*

- The base case is $V = 1$. What is the only thing a connected simple graph with one vertex can be? Check that the base case is satisfied.

- State the inductive hypothesis. (Be extremely careful with your wording!)

- Finish the induction. (Hint: Use Problem 11 to find a leaf. What can we say about the leaf and about the rest of the tree?)

4 Bonus Section: Hamiltonian Paths

Definition 5 *A path in a graph G is called* ***Hamiltonian*** *if it visits each vertex exactly once.*

One interesting case occurs if we look at a map of the United States. We can it into a graph by letting each state be a vertex, and connecting them if they share a border (corners, such as Utah and New Mexico, do not count). By excluding Alaska and Hawaii, we obtain a connected graph, and of course this is simple (check this!).

Problem 13 *Show that there is no Hamiltonian path on this graph which starts or ends at New York.*

Problem 14 *Suppose a Hamiltonian path starts in New England. Apart from New York, there is one other state where it cannot end in. Find it.*