Week 2: Cauchy induction and inequalities

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For this worksheet, let $P(n)$ be the statement: For all nonnegative $x_1, \ldots, x_n$, one has

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Problem 1 (AM-GM for 2 variables)**.**
Formulate and prove $P(1)$ and $P(2)$.
Remark: AM stands for *Arithmetic Means*, and GM stands for *Geometric Means.*
Hint: consider $(\sqrt{x_1} - \sqrt{x_2})^2$.

Problem 2.
Prove that $x + 1/x \geq 2$ for positive x.

Problem 3.
a) If $x + y = 6$, what is the biggest possible value for xy?

b) If $xy = 9$, what is the smallest possible value for $x + y$?

Problem 4.
a) Prove that for every nonnegative x, y and z, one has

$$x^2 + y^2 + z^2 \geq xy + xz + yz.$$

b) * Prove that for every nonnegative x, y and z, one has

$$x^3 + y^3 + z^3 \geq 3xyz.$$

Prove $P(3)$ from here.
Hint: Take the difference of LHS and RHS and factor out $(x + y + z)$.

It turns out that proving general $P(n)$ and is not that easy, and Cauchy proved $P(n)$ by using the following trick which is now known as Cauchy induction.

Problem 5.
Prove $P(n) \implies P(2n)$.
Hint: consider $\frac{a_1 + \cdots + a_n}{n}$ and $\frac{a_{n+1} + \cdots + a_{2n}}{n}$

Problem 6.
Prove $P(n) \implies P(n-1)$.
Hint: Let $a_n = \frac{a_1 + \cdots + a_{n-1}}{n-1}$

Problem 7.
Prove that $P(n)$ is true for every n.

We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ is *midpoint-convex* if

$$f\Big(\frac{x+y}{2}\Big) \;\leq\; \frac{f(x)+f(y)}{2} \qquad \text{for all } x,\, y \text{ in } \mathbb{R}.$$

Problem 8.
Use Cauchy induction to prove that, if f is midpoint convex, then

$$f\Big(\frac{x_1+...+x_n}{n}\Big) \;\leq\; \frac{f(x_1)+\cdots+f(x_n)}{n} \qquad \text{for all } n \geq 1 \text{ and } x_1,\, \ldots,\, x_n \text{ in } \mathbb{R}.$$

a) Do the $n \to 2n$ step;

b) Do the $n \to n-1$ step.

Problem 9.
a) Prove that the function $f(x) = x^2$ is midpoint-concave.

b) For positive $x_1, \ldots, x_n$ use the previous problem to prove the AM-QM inequality:

$$\frac{x_1+x_2+\cdots+x_n}{n} \leq \sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}}.$$

Remark: QM stands for *Quadratic Mean.*