ORMC Olympiad Group
Winter: Week 2
Modular Arithmetic II and Fermat's Thm

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Problems

1. Let $n > 1$ be a positive integer, and a, b are integers such that $(a, n) = 1$. Prove that there is an x in modulo n such that $ax \equiv b \mod n$

2. Using the problem 1, show that if $(a, b) = 1$, then there are integers x, y such that $ax + by = 1$

3. Prove *Fermat's Little Theorem.*

   Fermat's Thm Let p be a prime number. Then for any integer n, $a^p \equiv a \mod p$

4. (AIME 1986) What is that largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

5. For how many positive integers n less than 500

   $$300 | n^3 - 30n^2 + 200$$

   holds?

6. Find the largest possible value of prime number p, if there exist an

integer n such that

$$\begin{aligned} p &\mid n^2+3 \\ p &\mid (n+1)^2+3 \end{aligned}$$

7. **(AMC12 2014B)** The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

8. Find the number of nine digit integers in the form

$$*231*312*$$

which are divisible by 37. Here $*$ represents a single digit, but all three can be different.

9. For any two digit integer $n = \overline{ab}$, define $S_n = 121314\ldots\overline{ab}$, so that all two digit numbers from 12 to $n = \overline{ab}$ are put together in order. For example, $S_{14} = 121314, S_{32} = 121314\ldots3132$. Find the smallest two digit number $n \geq 12$ so that $99|S_n$

10. Let p be a prime number, so that both of the numbers p^2+p+1 and $p+10$ are also primes. Find p.

11. Prove that among n numbers there are $1 \leq k \leq n$ numbers which their sum is divisible by n.

12. Find the remainder when $2^{107}+3^{107}+6^{107}$ divided by 109.

13. (JBMO 2008 - shortlist) Is it possible to arrange the numbers $1^1, 2^2, 3^3, \ldots, 2008^{2008}$ one after the other, in such a way that the obtained number is a perfect square? (Explain your answer.)

14. Find all positive integers (n, m, k) such that $k \geq 2$ such that

$$1! + 2! + \cdots + n! = m^k$$

15. **(JBMO-2007)** Let p be a prime number. Show that $7p+3^p-4$ is not a perfect square.