Week 2: Cauchy induction and inequalities

Nikita

For this worksheet, let $P(n)$ be the statement: For all nonnegative $x_1, \ldots, x_n$, one has

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n}.$$

**Problem 1** (AM-GM for 2 variables).
Formulate and prove $P(1)$ and $P(2)$.
Remark: AM stands for *Arithmetic Means*, and GM stands for *Geometric Means*.
*Hint:* consider $(\sqrt{x_1} - \sqrt{x_2})^2$.

**Problem 2.**
Prove that $x + 1/x \geq 2$ for positive $x$.

**Problem 3.**
\(a\) If $x + y = 6$, what is the biggest possible value for $xy$?

\(b\) If $xy = 9$, what is the smallest possible value for $x + y$?

**Problem 4.**
\(a\) Prove that for every nonnegative $x, y$ and $z$, one has

$$x^2 + y^2 + z^2 \geq xy + xz + yz.$$

\(b\) * Prove that for every nonnegative $x, y$ and $z$, one has

$$x^3 + y^3 + z^3 \geq 3xyz.$$

Prove $P(3)$ from here.
*Hint:* Take the difference of LHS and RHS and factor out $(x + y + z)$.

It turns out that proving general $P(n)$ and is not that easy, and Cauchy proved $P(n)$ by using the following trick which is now known as Cauchy induction.

**Problem 5.**
Prove $P(n) \implies P(2n)$.
*Hint:* consider $\frac{a_1 + \cdots + a_n}{n}$ and $\frac{a_{n+1} + \cdots + a_{2n}}{n}$

**Problem 6.**
Prove $P(n) \implies P(n - 1)$.
*Hint:* Let $a_n = \frac{a_1 + \cdots + a_{n-1}}{n-1}$.
Problem 7.
Prove that $P(n)$ is true for every $n$.

We say that a function $f$ from $\mathbb{R}$ to $\mathbb{R}$ is midpoint convex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y$ in $\mathbb{R}$.

Problem 8.
Use Cauchy induction to prove that, if $f$ is midpoint convex, then

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + \ldots + f(x_n)}{n}$$

for all $n \geq 1$ and $x_1, \ldots, x_n$ in $\mathbb{R}$.

a) Do the $n \rightarrow 2n$ step;

b) Do the $n \rightarrow n - 1$ step.

Problem 9.
a) Prove that the function $f(x) = x^2$ is midpoint-concave.

b) For positive $x_1, \ldots, x_n$ use the previous problem to prove the AM-QM inequality:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}}.$$

Remark: QM stands for Quadratic Mean.