ORMC Olympiad Group
Winter: Week 1
Modular Arithmetic I and Divisibility Rules

Osman Akar

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Definition 0.1. Equivalency Let $n > 1, a, b$ be integers. We write

$$a \equiv b \mod n$$

iff n divides $a - b$. We say a is *equivalent* to b in modulo n.

Modular equivalence is a useful tool, because it accepts addition, subtraction and multiplication. It also accepts division for some cases, but we must need to be careful. More precisely, modular equivalence obeys the following rules:

Definition 0.2. Basic Properties of Modular Equivalency Let $n > 1, m \geq 0, a_1, a_2, b_1, b_2, c$ be integers. Assume $a_1 \equiv a_2 \mod n$ and $b_1 \equiv b_2 \mod n$. Then following holds:

1. $a_1 \pm b_1 \equiv a_2 \pm b_2 \mod n$
2. $a_1 \cdot c \equiv a_2 \cdot c \mod n$
3. $a_1 \cdot b_1 \equiv a_2 \cdot b_2 \mod n$
4. $a_1^m \equiv a_2^m \mod n$

Example 1. Numerical Examples

- $1 \equiv 14 \equiv -12 \equiv 131 \mod 13$

- $-17 \equiv -2 \equiv 13 \mod 15$
- $10^n \equiv 1 \mod 9$ for any $n \geq 0$. Note that this is used in divisibility rule for 9
- $10^n \equiv 1 \mod 11$ for any even $n \geq 0$, and $10^n \equiv -1 \mod 11$ for any odd $n \geq 0$. Note that this is used in divisibility rule for 11

Remark 0.1. Remark When we use modulo, it is important to remember all the time that we are the number represents congruence classes. E.g., when we are in modulo 12, the numbers -2, 10, 118, -62 does not have any difference. They all represent the congruence class $12k + 10$.

Example 2. Divisibility Rules in Modular Arithmetic Notation

- For any 3 digit integer $\overline{abc} \equiv a + b + c \mod 9$
- For any n-digit integer $\overline{a_1 a_2 \ldots a_n} \equiv a_1 + a_2 + \cdots + a_n \mod 9$
- For any 4 digit integer $\overline{abcd} \equiv -a + b - c + d \mod 11$
- For any n-digit integer $\overline{a_1 a_2 \ldots a_n} \equiv (-1)^{n-1} a_1 + (-1)^{n-2} a_2 + \cdots - a_{n-1} + a_n \mod 11$
- For any 6 digit integer $\overline{abcdef} \equiv -2a - 3b - c + 2d + 3e + f \mod 7$
- For any 8 digit integer $\overline{abcdefgh} \equiv +\overline{ab} + \overline{cd} + \overline{ef} + \overline{gh} \mod 99$
- For any 8 digit integer $\overline{abcdefgh} \equiv -\overline{ab} + \overline{cd} - \overline{ef} + \overline{gh} \mod 101$

Problems

1. **(AMC12 2008A-modified)** Let $k = 2022^3 + 3^{2022}$. What is the units digit of $k^3 + 3^k$?
2. Find digits a, b if $101 \mid 2a19177 6b9$.
   HINT: First try to create a divisibility rule for 101.
3. Find the last 2 nonzero digits of 20!.
4. For how many positive integers n less than 500

$$n^2 \equiv 6n + 66 \mod 75$$

holds?

5. **(HMMT Guts 2004)** Find all positive integer solutions (m, n) to the following equation:
$$m^2 = 1! + 2! + \cdots + n!$$

6. **(TNMO-FR 2018)** Let $x_0 = 2018$, and for $n \geq 1$ $x_n = x_{n-1} - 12$ or $x_n = 9x_{n-1} - 4$. For which of the fallowing values could be x_n for some n.

   (A) 100 **(B)** 10^{100} **(C)** 2018^{100} **(D)** $2018^{2018} - 2018$ (E) None

7. Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

8. (AIME 1986) In a parlor game, the magician asks one of the participants to think of a three digit number (abc) where a, b, and c represent digits in base 10 in the order indicated. The magician then asks this person to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine the (abc) if $N = 3194$.

9. For how many integer pairs (a, b) with boundaries $1 \leq a \leq 17$, $1 \leq b \leq 17$
$$17 | 1 + 4a + 5b + 3ab?$$

10. Find the largest possible value of prime number p, if there exist an integer n such that
$$\begin{aligned} p &\mid n^2 + 1 \\ p &\mid (n+1)^2 + 8 \end{aligned}$$

11. Find the number of 6 digits integers whose digits are chosen from $1, 2, 3, 4, 5$ and it is divisible by 101.

12. Find all positive integers (n, m, k) such that $k \geq 2$ such that
$$1! + 2! + \cdots + n! = m^k$$

13. Prove *Fermat's Little Theorem.*