Oleg Gleizer LAMC

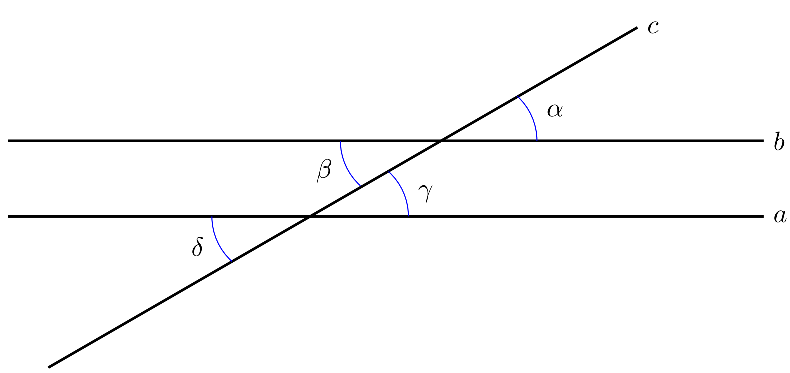
Introduction to Geometry Lesson 4

Parallel lines

Problem 1 *Recall the definition of vertical angles and prove that vertical angles are congruent.*

Two straight lines in the Euclidean plane are called *parallel* if they do not meet.

Proposition 1 *If straight lines a and b are parallel and a straight line c intersect them as shown on the picture below, then the angles α, β, γ, and δ are congruent.*



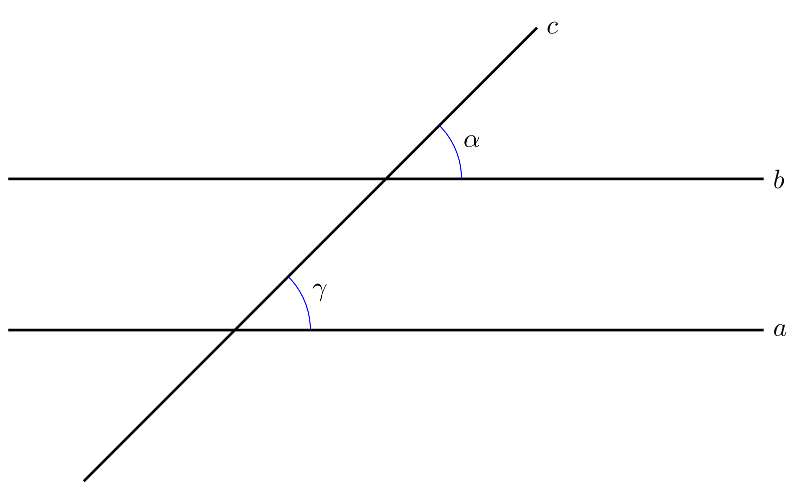
Problem 2 *Prove that $\alpha \cong \beta$ and $\gamma \cong \delta$.*

To complete the proof of Proposition 1, one needs to show that $\alpha \cong \gamma$. We are not going to do it at this time.

The following proposition is opposite to Proposition 1.

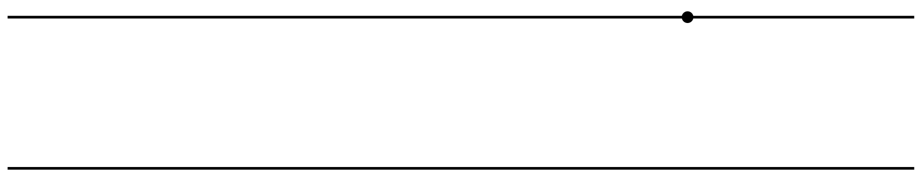
Proposition 2 *If two distinct straight lines in the Euclidean plane form the angles of equal size with a third straight line in the plane, then they are parallel.*

In other words, to check that the lines a and b on the picture below have no common point, you don't need to travel to infinity. All you need to do is to measure the angles α and γ. If $\alpha = \gamma$, then a is parallel to b.



We are not going to prove Proposition 2 either, but we are going to put it to use right away.

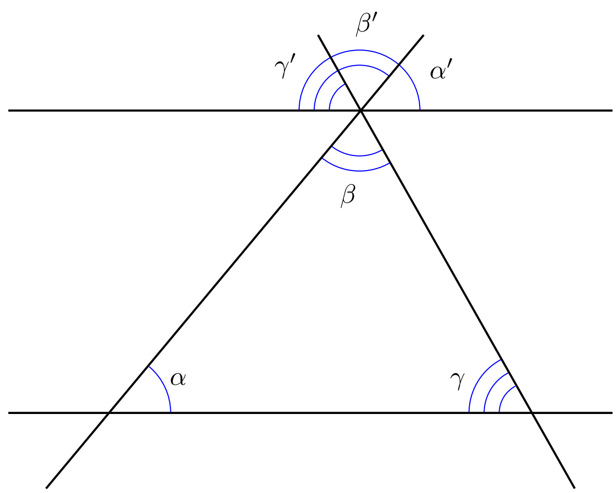
Postulate 5, Euclid - Playfair For any straight line in the (Euclidean) plane and for any point away from it, there exists a unique straight line that passes through the point and is parallel to the original line.



Problem 3 *Use a compass and a ruler to draw a straight line parallel to the one given below and passing through the given point not lying on the original straight line.*

Theorem 1 *The angles of any triangle in the Euclidean plane add up to a straight angle.*

Problem 4 *Use the below picture to prove the theorem.*



Problem 5 *Construct a* 60° *angle in the space below. Use a compass and a straightedge only, do not use a protractor.*

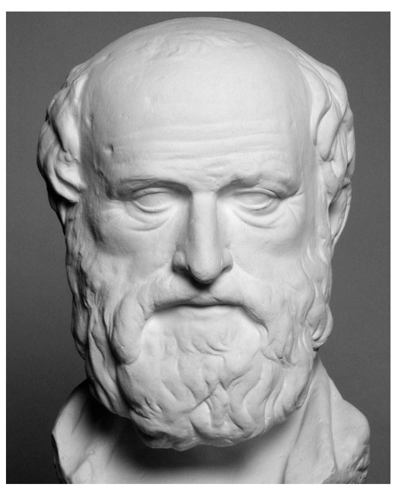
Theorem 2 *The angles of any quadrilateral in the Euclidean plane add up to* 360°*.*

Problem 6 *Prove Theorem 2*

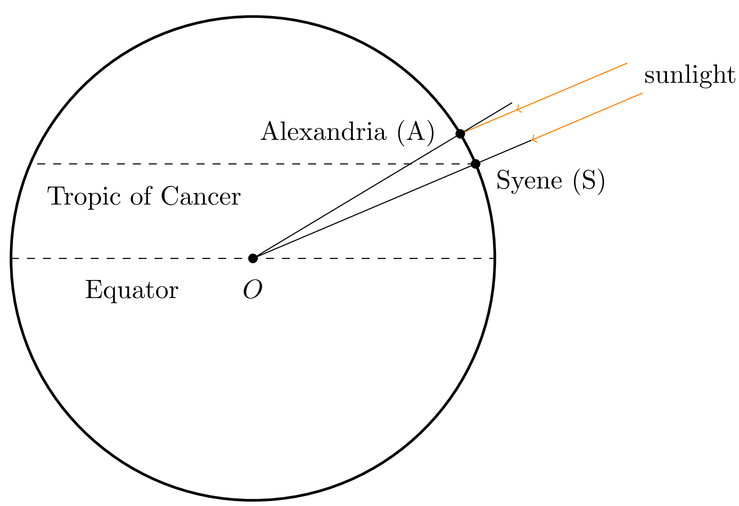
Problem 7 *We have defined a square as a quadrilateral with four sides of equal length and four right angles. Can we re-define a square as a quad with four congruent sides and angles? Why or why not?*

How to measure the radius of the Earth with a camel and a stick

Eratosthenes, the first man to measure the Earth, was born in 276 BC in Cyrene, a polis founded by the Greeks in 630 BC, now a city in the country of Libya. Appointed the chief librarian of the Great Library of Alexandria, the most famous research and teaching institution of the ancient world, the distinguished mathematician, astronomer and geographer was also renown as a music theorist, poet, and athlete.



The ancient city of Syene, currently Aswan in Egypt, is located on the Tropic of Cancer exactly south of Alexandria. If we stick a pole vertically in the ground (so that its lower end points to the center of the Earth) in Syene, the shadow the pole casts will disappear at noon of the Summer solstice day. Since Alexandria lies North of the tropic, the shadow of a similar vertically stuck pole will not disappear at this moment of time. The shadow and the pole will form the legs of a right triangle giving us a chance to measure its angle opposite to the ground. Thanks to Proposition 1 (well known to Eratosthenes), the angle will be equal to the angle between the pole and the sunlight as well as to the angle AOS between the Alexandria and Syene poles.



Eratosthenes also knew that the length l of a circumference was related to its radius R via the formula $l = 2\pi R$. The Greeks did not know how to find the value of π with any desired precision as we know nowadays. However, the approximation

$$\pi \approx \frac{22}{7}$$

was known to the classics and was precise enough for the purpose.

Please compare:

$$\begin{aligned} \pi &= 3.14159\ldots \\ \frac{22}{7} &= 3.14285\ldots \end{aligned}$$

Once we know the angle between the rays OA and OS, all we need to estimate the radius R of the Earth is the length l of the arc AS. Indeed,

$$\frac{l}{2\pi R} = \frac{\angle AOS}{360^\circ}. \tag{1}$$

The story has it that Eratosthenes took a camelback journey from Alexandria to Syene, counting the camel's steps on the way. To figure out l, he multiplied the number of the steps by the average step length. Since 1 is equivalent to the equation

$$R = \frac{360^\circ}{\angle AOS}\,\frac{l}{2\pi},$$

he proceeded to calculate the (average) radius of our planet[1] with an error of only 2%!

[1] The Earth is not a perfect sphere. The distance from its center to the surface points ranges from 6,353 km at the poles to 6,384 km at the Equator.

Problem 8 *The average radius of the Earth is 6,371 km. Imagine that our planet is a perfect sphere of this radius, having a perfectly smooth surface with no seas, mountains, etc. Further imagine a rope stretched tightly around the Earth's equator. They add a 1-meter-long piece to the rope and stretch it into a perfect circumference around the planet. Would a cat be able squeeze in between the rope and the Earth? Why or why not?*

Problem 9 *How fast is a point on the Equator moving, in kilometres per hour, due to the Earth's rotation? How can we possibly utilize this motion?*

Self-test questions

• How many congruent angles forms a straight line intersecting two parallel lines in the Euclidean plane? Draw a picture to answer.

• How to check if two straight lines in the Euclidean plane are parallel?

• What does the 5th postulate of Euclid (as formulated by Playfair) postulate?

• How to draw a straight line parallel to the given one and passing through a given point outside of the original line with a compass and a ruler?

• Why is the sum of the angles of any triangle in the Euclidean plane equal to 180° ?

• Why is the sum of the angles of any quadrilateral in the Euclidean plane equal to 360° ?

• Who was Eratosthenes? When did he live and where?

• What formula relates the radius and length of a circumference?

• How did Eratosthenes measure the radius of the Earth?

• Why is the main US space vehicles' launching cite located in Cape Canaveral, FL?