FALL 2021

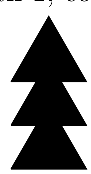
OLGA RADKO MATH CIRCLE
ADVANCED 2
DECEMBER 5TH, 2021

1. Merry Christmas and a Happy New Year

Problem 1.1. In a gift-giving tradition known as Secret Santa, n people each buy a gift for exactly one other person (not themselves), in such a way that every person receives exactly one gift.

(1) (2 points) Suppose there are 4 people. In how many ways can this be done?
(2) (3 points) Give a formula for the general case in terms of n. (The formula can use summations, recursion, etc. if necessary.)

Problem 1.2 (2 points)**.** A 2D Christmas tree is constructed by stacking 3 identical equilateral triangles, such that a vertex of each triangle is located at the centroid of triangle above it, as shown. If each triangle has side length 1, compute the area of the figure.



Problem 1.3 (2 points)**.** Consider the following:
I don't want a lot for Christmas
There is just one thing I need
I know that $uv = 2u + v + 4$
And that u, v are in $\mathbb{Z}$
I just want that $u^2 - v^2$
Is exactly twenty-four
Make my wish come true
All I want for Christmas is u

2. Miscellaneous problems part 1

Problem 2.1 (2 points)**.** Each of the vertices of the base of a triangle is connected by straight lines to 50 points on the side opposite it. Into how many parts do these 100 lines divide the interior of the triangle?

Problem 2.2 (2 points)**.** Prove that if x is a nonzero real number and $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer for all integers n.

Problem 2.3 (2 points)**.** You are presented with two fuses (lengths of string), each of which will burn for exactly 1 minute, but not uniformly along its length. Find a way to measure 45 seconds using these two fuses.

Problem 2.4 (2 points)**.** How many ways are there to write the numbers 0 through 9 in a row, such that each number other than the left-most is within one of some number to the left of it?

Problem 2.5 (2 points)**.** Find a polynomial with integer coefficients and of degree 4, such that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial.

Problem 2.6 (2 points)**.** Let n be a positive integer. Find all the real roots to the equation

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{3x}}}}}}_{n\text{ radicals}} \;=\; x.$$

(Note that all roots are considered nonnegative.)

Problem 2.7 (2 points)**.** Find the smallest positive integer $k>10$ where $\sqrt{\frac{k!(k+1)!}{2}}$ is an integer.

Problem 2.8 (3 points)**.** Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

Problem 2.9 (3 points)**.** If $ABCD$ is convex 4-gon, the lengths of AB, BC, CD, DA, AC, BD are rational. If AC, BD intersect at O, prove that the length of AO is also rational.

3. Miscelleneous problems part 2

Problem 3.1 (3 points)**.** The number 6 is called a *perfect number* because its proper divisors sum up to itself: $1+2+3=6$. 28 is another such perfect number, as $1+2+4+7+14=28$. Find the smallest perfect number greater than 28.

Problem 3.2 (3 points)**.** Find $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Problem 3.3 (3 points)**.** You are challenged to play the following game: you start with \$1. Each round, you gain \$1 but then must return $\frac{2}{3}$ of your money. After playing an infinite number of rounds, how much money will you end up with?

Problem 3.4 (4 points)**.** In how many ways can a 100-dollar bill be changed into 1-, 2-, 5-, 10-, 20-, and 50-dollar bills?

Problem 3.5 (4 points)**.** There is an equation $x^3+ax^2+bx+c=0$ written on the black board. Peter first replaces one of a, b, c by a **nonzero** integer, then Lily replaces another letter by an integer, then Peter replaces the last letter by an integer. Help Peter to create a strategy such that, no matter what Lily does, Peter can make sure the 3 roots of the final equation (of x) are all integers.

Problem 3.6 (4 points)**.** If P is a convex polygon such that no triangle of area 1 can fit in it, prove that P fits in a triangle of area 4.

Problem 3.7 (5 points)**.** There are n distinct points lying on a circle and there is a chord connecting every two of them; assume no three chords intersect, find out the number of regions the disk is divided into by the chords?

Problem 3.8. (6 points) Given a connected graph, a **Hamiltonian path** from v to w is a path which starts at v, ends at w, and visits every vertex exactly once.
Consider the continental United States (i.e., without Hawaii and Alaska) and view it as a graph in the following manner:

(1) The vertices are the states
(2) Two vertices share an edge if the states share a border

Notice that Maine only borders one state, so any Hamiltonian path would have to either start or end in Maine. Suppose we start in Maine. New York is what's called a *bottleneck*; removing New York would disconnect the graph. Therefore, any Hamiltonian path which starts at Maine would have to end on the other side of New York (which we will call the Western US). An all knowing computer has verified that there are Hamiltonian paths ending at every single state west of New York **except one**.

Which state is the exception?