Math Circle Advanced 1

Fall quarter 2021 final game

Problem 0.1.
Find the number of solutions to the equation

$$|x| + |y| = 100.$$

Here the solutions (x, y) and (y, x) are considered to be different if $x \neq y$.

Problem 0.2.
Find all the solutions to the following equation:

$$|x+1| \;-\; |x| \;+\; 3|x-1| \;-\; 2|x-2| \;=\; x+2.$$

Problem 0.3.
Evaluate the following sum:

$$\binom{10}{0} \;+\; \frac{1}{2}\binom{10}{1} \;+\; \frac{1}{3}\binom{10}{2} \;+\; \ldots \;+\; \frac{1}{11}\binom{10}{10}.$$

Problem 0.4.
Find all natural numbers n such that $\frac{n+82}{2n-5}$ is a natural number.

Problem 0.5.
the quadratic equation $x^2 + mx + n$ has roots twice those of $x^2 + px + m$ and none of p, n, or m are zero. What is the value of n/p?

Problem 0.6.

Find all real numbers x such that the following is true:

$$\frac{1+\frac{1+\frac{1+x}{1-3x}}{1-3\frac{1+x}{1-3x}}}{1-3\frac{1+\frac{1+x}{1-3x}}{1-3\frac{1+x}{1-3x}}} = 1.$$

Problem 0.7.
How many different solutions in positive integers does the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10$$

have?

Problem 0.8.
Row a dice with 6 faces (number 1 to 6) twice. What is the probability that the product of two numbers you get is 12?

Problem 0.9.
What is the largest number of the following statements about a real number x, that can be true at the same time?

(i) $x^2 < -1$,

(ii) $0 < x^2 < 1$,

(iii) $x^2 > 1$,

(iv) $-1 < x < 0$,

(v) $0 < x < 1$,

(vi) $0 < x - x^2 < 1$.

Problem 0.10.
What is the maximal number of chess knights one can place on a chessboard without them attacking each other?

Problem 0.11.
Find a polynomial with integer coefficients and of degree 4, such that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial.

Problem 0.12.
Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is the value of $abcd$?

Problem 0.13.
How many ways are there to write the numbers 0 through 9 in a row, such that each number other than the left-most is within one of some number to the left of it?

Problem 0.14.
Asya and Vasya have three coins. On different sides of one coin, scissors and paper are depicted, on the sides of another coin, a rock and scissors, on the sides of the third, paper and rock. Scissors beats paper, paper beats rock and rock beats scissors. First Asya chooses a coin for herself, then Vasya, then they throw their coins and see who won (if the same thing fell out, then it?s a draw). They do this many times. Does Vasya have an opportunity to choose a coin so that the probability of his winning is higher than that of Asya?

Problem 0.15.
Let x be a nonzero real number such that $x+\frac{1}{x}=10$ is an integer. Compute $x^4+\frac{1}{x^4}$.

Problem 0.16.
Here is a quadratic equation $x^2+bx+c=0$. You randomly choose b and c from the set $\{1,2,3,4,5,6\}$. What is the chance that your equation has no real solutions.

Problem 0.17.
A and B shoot at the shooting range, but they only have one six-round revolver with

one round. They agreed to take turns randomly spinning the drum and firing. A begins. Find the probability that a shot will occur when A has the revolver.

Problem 0.18.
You have 5 envelopes and 5 different letters to send. Each envelope has the receiver's address written that corresponds to the exactly one of the five letters. You are in a rush during the day sending all the letters at once and randomly seal letters to envelopes. What is the probability that no letter is in its proper envelope?

Problem 0.19.
Let n be a positive integer. Find all the real roots to the equation

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\ldots+2\sqrt{x+2\sqrt{3x}}}}}}_{n \text{ radicals}} = x.$$

(Note that all roots are considered nonnegative.)

Problem 0.20.
You are presented with two fuses (lengths of string), each of which will burn for exactly 1 minute, but not uniformly along its length. Find a way to measure 45 seconds using these two fuses.

Problem 0.21.
The product of three consecutive ositive integers is 8 times their sum. What is the sum of their squares?

Problem 0.22.
Mailboxes numbered 1 to 50 stand in a row at an apartment building. When the first tenant arrives, she opens all the mailboxes. The second tenant then goes through and recloses all the even-numbered mailboxes; the third tenant changes the state of every mailboxes whose number is a multiple of 3. This continues until 50 tenants have passed through. Which mailboxes are now open?

Problem 0.23.
Alice, Bob, and Carol arrange a three-way duel. Alice is a poor shot, hitting her target only $\frac{1}{3}$ of the time on average. Bob is better, hitting his target $\frac{2}{3}$ of the time. Carol is a sure shot. They take turns shooting, first Alice, then Bob, then Carol, then back to Alice, and so on until only one is left. What is Alice's best course of action?

Problem 0.24.
The 2020–21 Candidates Tournament was an eight-player chess tournament to decide the challenger for the World Chess Championship 2021. In this tournament, each player played another player twice, once as white, and once as black. Ian Nepomniachtchi won the tournament with and earned the right to challenge the defending world champion, Magnus Carlsen. How many games were played in this tournament?

Problem 0.25.
Each of the vertices of the base of a triangle is connected by straight lines to 50 points on the side opposite it. Into how many parts do these 100 lines divide the interior of the triangle?

Problem 0.26.
Find the number of ways to place a white and a black king on a chessboard so that they don't attack each other.

Problem 0.27.
Find the number of ways to place 14 bishops on a chessboard so that they don't attack each other.

Problem 0.28.
What move can White make and NOT mate in one?