Olga Radko Math Circle Competition

Advanced 3

Week 5 Fall 2021

Each problem is worth 2 points unless otherwise noted. Questions that are marked as 4 points are generally harder.
Unless otherwise noted problems may be attempted as many times as you want without any penalty, however this may be changed at the discretion of the instructor.
In general problems with a numerical solution will be given full points for just the correct numerical value however this is up to the discretion of the instructor (especially for problems with easily guessed numerical answers).

1. Nimbers

Problem 1.1. Let's play multiplying Nim! Get some number of players and ask Aaron. You will go to a separate breakout room, and the game will be played over jamboard.
You will get to decide whether you want to go first or second. You get points only if you win.
If you need a reminder here is the link to the nimber's worksheet: Click here.

Problem 1.2. Consider playing Nim with a single pile but you are only allowed to remove a number of stones between 1-9 but not 5. What are the losing positions?

Problem 1.3. Find the losing positions in big Nim, which is like normal Nim except that you can only remove stones from the pile with the most stones.

2. FFT

Problem 2.1. Compute

$$\sum_{j=1}^{n} j^2$$

Problem 2.2. For each pair f, g determine if $f \in O(g)$.
Each part is worth 1 point, based only on whether the answer is correct. Only one attempt will be allowed. All parts must be attempted together.
a: $f(x) = x^2$, $g(x) = x^3$
b: $f(x) = \frac{e^x}{x^4}$, $g(x) = x^4$.
c: $f(x) = x^{\frac{1}{1+\log\log x}}$, $\log x$.

d: $f(x) = \frac{1}{e^{\frac{1}{x^2}}-1}$, $g(x) = x^2$.

Problem 2.3. (4 points) Consider the polynomial $p(z) = \sum_{k=0}^{n} \frac{1}{n-k+1} z^k$, where z is a complex number. Show that the roots of p are all in the closed unit disk.

Problem 2.4. Shapiro own a fleet of 99 trucks he uses for shipping. Each truck starts at rest at time $t = 0$ and after $t = 0$ each truck driver starts moving with constant acceleration (where each truck has a different acceleration).
Each truck carries some fixed number of pounds of cargo and Shapiro charges \$7 per pound per mile moved.
After 1 minute Shapiro has made \$3. After 3 minutes he has made \$21. How much money has he made in 10 minutes.

3. Voting Theory

Problem 3.1. Suppose that the following ballots are cast

$$(3) A > B > C > D > E$$
$$(2) D > E > C > A > B$$
$$(2) C > E > D > B > A$$
$$(1) C > D > A > B > E$$

Can you come up with a fair voting system to make E win?

4. Miscellaneous Problems

Problem 4.1. Find a polynomial with integer coefficients and of degree 4, such that $\sqrt{2}+\sqrt{3}$ is a root of the polynomial.

Problem 4.2. What are the last two digits of 3^{1234}?

Problem 4.3. Show that any collection of n integers has a non-empty subset whose sum is divisible by n.

Problem 4.4. Are there ten consecutive numbers so that each one is divisible by a perfect cube (besides 1).

Problem 4.5. Bob puts 100 coins (of various denominations) in a row on a table. Alice and Bob take turns each removing a coin from either end with Alice going first. Can Alice always guarantee to get more money then Bob?

Problem 4.6. Each part is worth two points.
Two dokemons do battle. Each dokemon starts with 100 health points. Each dokemon takes turns performing a move. Each move deals some damage to both the opposing dokemon and the dokemon casting the move but every move deals at least as much damage to the opposing dokemon as it causes to the dokemon casting. Both dokemon have the same set of moves. Moves can be used as many times as desired. A dokemon wins if its opponent faints (strictly) before it.
a: Show if the set of moves is finite then the second dokemon cannot force a win.
b: What if the move set is infinite?

Problem 4.7. a: Does $\sum_n \frac{1}{n}$ converge? (1 point)
b: What if we take the sum over only those n that don't have two adjacent 9's in there decimal representation? (2 points)

Problem 4.8. Prove by explicit bijection that the number of partitions of n into odd parts is equal to the number of partitions into distinct parts.
For example if $n = 6$ the partitions into distinct parts are $6, 1+5, 2+4, 1+2+3$ and the partitions into odd parts are $5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1$.

Problem 4.9. How many ways are there to write the numbers 0 through 9 in a row, such that each number other than the left-most is within one of some number to the left of it? e.g. 0123456789 and 987654321 counts, but 0123476589 is not.

Problem 4.10. Find the coefficients of x^{17} and x^{18} in $(1 + x^5 + x^7)^{20}$.

Problem 4.11. Bob wakes up in a prison cell. In his cell there is a very deep pool filled with marbles. The warden walks in and tells him the following.
Your cell will be guarded by one of two prison guards John or James. John has a magical ability to tell how many marbles are in the pool at any time, whereas James is just a normal person who is a very good liar. Every ten minutes your prison guard will enter your cell and tell you how many marbles are in the pool (if it is John then it will be the correct number and if it is James then he will make something up). After ten hours you will have to figure out whether your cell was being guarded by John or James.
Knowing that Bob has a (fair) coin in his back pocket, can you come up with a plan?
Note your plan needs to work even if James is an "adversary" which means that James will know exactly what your plan is. The only thing hidden from James is your coin flips.

Problem 4.12. Let n be a positive integer. Find all the real roots to the equation

$$\underbrace{\sqrt{x + 2\sqrt{x + 2\sqrt{x + \ldots + 2\sqrt{x + 2\sqrt{3x}}}}}}_{n \text{ radicals}} = x.$$

(Note that all roots are considered nonnegative.)

Problem 4.13. (4 points) A rectangle is cut up into (finitely many) small rectangles. Prove that if each of the small rectangles have an integer side then the original rectangle does too.

Problem 4.14. Each part is worth two points.
Imagine an infinite line of pots where each pot has some number of coins. Each second you pick a pot with two or more coins. You remove two coins from this pot and place them in the two adjacent pots (one in each pot). How many seconds till every pot has at most one coin?
a: Initially one pot has 100 coins and the rest are empty.
b: Initially one pot has 100 coins, the pot to its left has 101 coins and the rest are empty.
Hint: The result of Problem 2.1 might be helpful.

Problem 4.15. (4 points) Given a connected graph, a **Hamiltonian path** from v to w is a path which starts at v, ends at w, and visits every vertex exactly once.
Consider the continental United States (i.e., without Hawaii and Alaska) and view it as a graph in the following manner:

(1) The vertices are the states
(2) Two vertices share an edge if the states share a border

Notice that Maine only borders one state, so any Hamiltonian path would have to either start or end in Maine. Suppose we start in Maine. New York is what's called a *bottleneck*; removing New York would disconnect the graph. Therefore, any Hamiltonian path which starts at Maine would have to end on the other side of New York (which we will call the Western US). An all knowing computer has verified that there are Hamiltonian paths ending at every single state west of New York **except one**.
Which state is the exception?