HERE'S LOOKING AT EUCLID

MATH CIRCLE (BEGINNERS) 02/19/2012

You work at the **HopBot Testing Facility**, home of the world's premier hopping robot. A **HopBot-X** will start at 0 on the number line, and repeatedly hop X spaces. So for instance, a HopBot-3 will hop: 0, 3, 6, 9, 12, 15, etc.

There are different kinds of HopBot. The **SlopBot** is designed to feed pigs, so every time it hops, it drops some slop on top of the spot. The **MopBot** is for cleaning up; every time it hops, it mops up any slop it spots on the spot.

(1) You recently tested a SlopBot-52 and a SlopBot-76 (on the same number line), and both worked perfectly. Now you want to mop up the slop with a MopBot. What is the largest hop-setting X you can use, so that a single run of MopBot-X will clean up all the slop from BOTH the SlopBots?

(2) What is the best single MopBot-X (the one with the largest possible X) to clean up after a SlopBot-91 and a SlopBot-119?

(3) What is the best single MopBot-X to clean up slop from a SlopBot-54 and a SlopBot-90?

(4) What is the best single MopBot-X to clean up slop from a SlopBot-57 and a SlopBot-41?

(5) What is the best single MopBot-X to clean up slop from a SlopBot-35 and SlopBot-105?

(6) What is the best single MopBot-X to clean up slop from a SlopBot-24, SlopBot-45, and SlopBot-100?

(7) What is the best single MopBot-X to clean up slop from a SlopBot-88, SlopBot-120, and SlopBot-144?

(8) What is the best single MopBot-X to clean up slop from a SlopBot-2100, SlopBot-1232, and SlopBot-1820?

(9) What is $\gcd(2^5 3^2 5^3 7^1 13^2 17^4 23^1, 2^2 3^4 5^1 7^6 11^4 13^2 17^3)$? (Hint: For goodness' sake, *don't* try to multiply those numbers out–you won't be able to! Express your answer as a product of primes.)

(10) Sony is thinking of a number x such that $\gcd(x, 78) = 6$, $\gcd(x, 110) = 10$, and $\gcd(x, 105) = 15$. What is the smallest number x could be?

(11) $96 \equiv 6 \pmod{10}$. Compute

(a) $\gcd(96, 10) =$

(b) $\gcd(10, 6) =$

(12) $77 \equiv 12 \pmod{13}$. Compute

(a) $\gcd(77, 13) =$

(b) $\gcd(13, 12) =$

(13) $220 \equiv 20 \pmod{40}$. Compute

(a) $\gcd(220, 40) =$

(b) $\gcd(40, 20) =$

The idea of Euclid's Algorithm is that if $r \equiv a \pmod{b}$ (remember this means r is the remainder when you divide a by b), then $\gcd(a, b) = \gcd(b, r)$. To do the Euclidean Algorithm, start by writing the result when you divide a by b with remainder, and repeat the process:

$$\begin{aligned}
a &= q_1 b &+& \; r_1 \qquad (0 < r_1 < b)\\
b &= q_2 r_1 &+& \; r_2 \qquad (0 < r_2 < r_1)\\
r_1 &= q_3 r_2 &+& \; r_3 \qquad (0 < r_3 < r_2)\\
r_2 &= q_4 r_3 &+& \; r_4 \qquad (0 < r_4 < r_3)\\
&\vdots\\
r_{n-2} &= q_n r_{n-1} &+& \; r_n \qquad (0 < r_n < r_{n-1})\\
r_{n-1} &= q_{n+1} r_n &+& \; 0
\end{aligned}$$

So you stop when you get a remainder of 0, and then you've found $\gcd(a, b) = r_n$—it's simply the last nonzero remainder (the one circled above).

Here's an example using Euclidean Algorithm find $\gcd(76, 32)$ (so in this case $a = 76$, $b = 32$):

$$\begin{aligned}
76 &= 2 \cdot 32 + 12\\
32 &= 2 \cdot 12 + 8\\
12 &= 1 \cdot 8 + 4\\
8 &= 2 \cdot 4 + 0
\end{aligned}$$

So the last nonzero remainder is 4, and you can check that this is the correct gcd of 76 and 32.

(14) Use the Euclidean Algorithm to find $\gcd(75, 40)$.

(15) Use the Euclidean Algorithm to find $\gcd(245, 193)$.

(16) If p, q, and r are three different prime numbers, what is $\gcd(pq, pr)$?

(17) Here's a secret: $1591 = pq$ and $1739 = pr$ for some three different prime numbers p, q, r. Find p using the Euclidean Algorithm.