# ORMC Intermediate 2 Competition

Kevin Li

December 2021

Welcome to the last week of this quarter's math circle! The tournament will consist of 2 teams, and 19 questions. The questions do NOT have to be done in order. Each team will get 2 submissions per question where you will tell an instructor your proposed solution. You will be awarded 1 point for a correct solution, 0.5 points for a partial solution, and 0 points for a mostly incorrect solution. Good luck!

## 1 Binary Arithmetic, Boolean Algebra, Logic

**Problem 1.** Recall the Disjunctive Normal Form (DNF) and the Full Disjunctive Normal Form (FDNF). Give an example of the expression

$$AB + AC$$

with english statements and compute the FDNF.

**Problem 2.** Compute the following products (0.5 points each):

$$10110_2 * 101_2$$

$$10010_2 * 1111_2$$

## 2 Combinatorics, Permutations

**Problem 3.** Suppose you color each of the vertices of a regular heptagon either black or red. We say two colorings are the same under rotational symmetry if one coloring can be achieved by rotating another coloring. How many distinct colorings under rotational symmetry are there?

**Problem 4.** How many perfect squares divide the number

$$4! * 5! * 6!$$

**Problem 5.** How many ways can you distribute 9 quarters to 4 students?

**Problem 6.** Recall the definition of injective, surjective, and bijective. If $f : X \to Y$, $g : Y \to Z$ are bijective, show that $(g \circ f) : X \to Z$ is bijective.
(Hint: Try to prove the statement if 'bijective' was replaced by 'injective' and 'surjective' separately.)

**Problem 7.** Let $n$ be an even natural number. How many permutations $f : [n] \to [n]$ exist such that no two consecutive integers both map to even numbers? (For example if f(3)=2 then f(2) and f(4) must be odd)

## 3 Complex Numbers, Geometry

**Problem 8.** In triangle $ABC$, $m\angle A = 90°$, $AC = 1$, and $AB = 3$. Point $D$ lies on ray $\overrightarrow{AC}$ such that $m\angle DBC = 2m\angle CBA$. Compute the area of triangle $ABD$.

**Problem 9.** Find all of the solutions of the following complex equations and sketch them in the Argand plane (0.5 points each)

$$z^7 = 3$$

$$z^5 + 1 = 0$$

**Problem 10.** Let $L$ be a line and $A, B, C$ be three collinear points lying entirely on one side of $L$. Show that their reflections $A', B', C'$ across $L$ are also collinear.

**Problem 11.** Show that the midpoints of any quadrilateral form a parallelogram.

**Problem 12.** Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6$, and $DA = 7$. Let $A_1$ and $C_1$ be the feet of the perpendiculars from $A$ and $C$, respectively, to line $BD$, and let $B_1$ and $D_1$ be the feet of the perpendiculars from $B$ and $D$, respectively, to line $AC$. Find the perimeter of quadrilateral $A_1B_1C_1D_1$.

## 4 Other Topics

**Problem 13.** Prove, using induction, that $n < 2^n$ for all natural numbers n.

**Problem 14.** Prove, using induction, that $1 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

**Problem 15.** Lockers numbered 1 to 100 stand in a row at a school building. When the first student arrives, she opens all the lockers. The second student then goes through and recloses all the even-numbered lockers. The third student changes every locker whose number is a multiple of 3. This continues until 100 students have passed through. How many lockers are now open?

**Problem 16.** Are there any integer solutions to the equation $x^2 + 3y^2 = 360$? Prove it.

**Problem 17.** Let $k = 2021^3 + 3^{2021}$. What is the units digit of $k^3 + 3^k$?

**Problem 18.** Find a sequence of 38 consecutive natural numbers such that none of their digit sums is divisible by 11.

**Problem 19.** (Final Problem) A calculator is programmed to calculate a fixed polynomial $P$ with positive integer coefficients and unknown degree. When you input a positive integer $n$, it will display the value $P(n)$, which counts as one step. You are allowed to repeat this process. Try to figure out the polynomial $P$ in as few steps as possible.

When you submit your answer, the instructor will have two polynomials prepared in advance. You will be asked to guess the polynomial after 5 steps (for practical concerns), and if correct you will get $(6-n)/4$ (where $n$ is the number of guesses) or 1 point, whichever is lower, for each polynomial, for a total of up to two points.