Combinatorics and Permutations 2

Kevin Li

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1 Cycles and Inverse Functions

Last week, we explored properties of functions. We will be focusing on **bijective** functions, $f : [n] \to [n]$. Last time we showed that the number of such functions is **exactly** the number of permutations of $[n]$. One important property of bijective functions is that they are **invertible**. In other words, there is a well-defined way to go backwards.

Example 1. Consider the function $f : [4] \to [4]$ defined by

$$f(1) = 3, f(2) = 1, f(3) = 2, f(4) = 4$$

Remember, to write f as the product of cycles, we have to track the movement of each element.

$$1 \to 3 \to 2 \to 1, 4 \to 4$$

Thus we can write $f = (132)(4) = (132)$ since when we return to the element that we started with, we know the pattern will continue.

Problem 1. Let $f : [7] \to [7]$ be defined by

$$f(1) = 6, f(2) = 5, f(3) = 7, f(4) = 3, f(5) = 1, f(6) = 2, f(7) = 4$$

Write f as the product of cycles.

Problem 2. Now, let's work backwards. Let $f : [5] \to [5]$ be defined by $f(x) = (143)(25)$. Find where f maps each element of [5].

Problem 3. Using the same f as in problem 2, how would we define the inverse function f^{-1}? Think about where f sends each element in [5] and try to work backwards.

Example 2. How would we figure out where a function sends each element if we have a more complicated product of cycles? Let's suppose we have $f : [5] \to [5]$,

$$f(x) = (1532)(14)(35)$$

Notice so far that all of our cycles before this have only contained unique elements. No two different cycles have shared an element. All of the previous functions have be written as the product of **disjoint cycles**. Since 5 shows up in two of the cycles of f, (1532) and (35), this is **not** the product of disjoint cycles. Could we represent f as a product of disjoint cycles? Let's track the movement of each element.
Notice how our f has 3 cycles, namely (1532), (14), and (35). To track the movement of 1, we have to consider how the cycles interact with each other. Let's think about these 3 cycles as 3 unique maps.

Definition 3. Suppose $f : X \to Y$, $g : Y \to Z$. We define the **composition** g of f to be $(g \circ f)(x) = g(f(x))$. We first apply f to x, and then apply g to $f(x)$.

Problem 4. Suppose $f : X \to Y$, $g : Y \to Z$. What is the domain and codomain of $g \circ f$?

Problem 5. Let $f : \{1, 2, 3\} \to \{4, 5, 6\}, g : \{4, 5, 6\} \to \{1, 2, 3\}$,

$$f(1) = 5, f(2) = 4, f(3) = 6, g(4) = 3, g(5) = 1, g(6) = 2$$

Compute

$$(g \circ f)(1), (g \circ f)(2), (g \circ f)(3)$$

$$(f \circ g)(4), (f \circ g)(5), (f \circ g)(6)$$

Continuing the above example, lets consider three separate functions.

$$f_1(x) = (1532), f_2(x) = (14), f_3(x) = (35)$$

As a convention, we read the product of cycles from **left to right**. So we would first see where f_1 sends the element 1. We want to see where the **first** cycle sends 1. So computing,

$$f_1(1) = 5$$

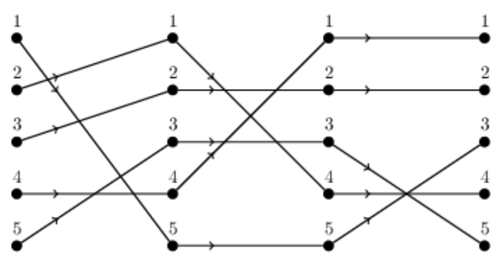
Then, we want to see where the **second** cycle sends this new element,

$$f_2(f_1(1)) = f_2(5) = 5$$

since $(14) = (14)(2)(3)(5)$. Lastly, applying the **third** cycle,

$$f_3(f_2(f_1(1)) = f_3(f_2(5)) = f_3(5) = 3$$

Thus, we have the map $1 \mapsto 5 \mapsto 5 \mapsto 3$. There are 3 different cycles, so every element has to be mapped 3 times. If we do the same thing for each of the elements, we can see that the following chart can describe the movement of each element.



So overall, we see that the function $f(x) = (1532)(14)(35)$ is the same thing as

$$f(1) = 3, f(2) = 4, , f(3) = 2, f(4) = 1, f(5) = 5$$

Problem 6. Write the above f as the product of **disjoint** cycles.

Problem 7. (CHALLENGE)
Let $f : [n] \to [n]$ be a bijection. Prove f can be written as the product of **disjoint cycles**.

It might help to consider the following definition.

Definition 4. Let $f : [n] \to [n]$ be a bijection. A subset of the codomain (or image) $f([n])$ is **cyclic** if it can be written as $\{x, f(x), f^2(x), ..., f^k(x)\}$ for some $k \in \mathbb{N}$ and for some $x \in [n]$.

Note that cycles are cyclic.

Problem 8. Define an inverse f^{-1} of $f : [5] \to [5], f(x) = (1324)(5)$. Write f^{-1} in terms of where it maps each element, and as a product of disjoint cycles. Notice the relationship between the cycles of f and the cycles of f^{-1}.

Definition 5. Let $f : X \to Y$ be a bijective function. We say $g : Y \to X$ is the **inverse** of f if for all $x \in X$, $f(x) = y \iff g(y) = x$. In other words, $(g \circ f)(x) = x$ for all $x \in X$ and $(f \circ g)(y) = y$ for all $y \in Y$. We usually denote such g by f^{-1}?

Problem 9. Is every bijective function invertible? Why or why not?

Problem 10. Let $f : X \to Y, g : Y \to Z$. Show that the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$.

Problem 11. Let $f : X \to Y, g : Y \to Z$, $h : Z \to W$. Show that the inverse of $h \circ g \circ f$ is $f^{-1} \circ g^{-1} \circ h^{-1}$

Problem 12. Now, using problem 11 and that the relationship between the cycles of a function and its inverse (from problem 8), find the inverse of $f : [5] \to [5]$,

$$f(x) = (1532)(14)(35)$$

(**Hint:** remember, we can think of f as the product of 3 different cycles which are written as functions with disjoint cycles: $f_1(x) = (1532), f_2(x) = (14), f_3(x) = (35)$. How would we write f as a composition of f_1, f_2, f_3?)

To end the topic of permutations, I will give you a few definitions for further other interesting ideas to explore on your own.

Definition 6. We denote the collection of all bijections $f : [n] \to [n]$ by the set S_n.

Definition 7. Let $f : [n] \to [n]$ bijection. We call $g : [n] \to [n]$ bijection a **conjugation** of f if there exists a $p : [n] \to [n]$ bijection such that $(p \circ f \circ p^{-1})(x) = g(x)$ for all $x \in [n]$.

There are a lot of interesting properties of conjugations. If you are interested in learning about conjugacy classes, I highly encourage you to explore some properties by picking a small $n \in \mathbb{N}$ and seeing what you can come up with. Can you guess, based on properties of cycles, what types of functions would be conjugates of one another?