The Complex Numbers and Geometry II

Yan Tao

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1 De Moivre's Theorem

Recall from last week that we have the two different forms of a complex number: $z = a + bi$ the rectangular form, and $z = (r, \theta)$ the polar form, which are related by the following equations

$$a = r\cos(\theta) \text{ and } b = r\sin(\theta)$$
$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}(\frac{b}{a})$$

We also proved the following trig identities to help us deal with sine and cosine functions:

$$\cos(\frac{\pi}{2} - \theta) = \sin(\theta) \text{ and } \sin(\frac{\pi}{2} - \theta) = \cos(\theta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \text{ and } \cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Let us put it all together this week. First, we introduce *proof by induction:*

Theorem 1 *Let $P(1), P(2), ...$ be statements. Suppose that*

- *(The Base Case) $P(1)$ is true*
- *(The Inductive Step) Assuming that $P(k)$ is true, $P(k+1)$ is then also true.*

Then $P(n)$ is true for all $n = 1, 2, ...$

Definition 1 *In the inductive step, when we assumed that $P(k)$ is true, that assumption is called the* ***inductive hypothesis*** *or the* ***hypothesis of induction****.*

Our first example of a proof by induction will be one of the most powerful theorems about complex numbers, called **de Moivre's Theorem**.

Theorem 2 *(de Moivre) For all $n = 1, 2, ...$*

$$(cos(n\theta) + i\sin(n\theta)) = (\cos(\theta) + i\sin(\theta))^n$$

Problem 1 *Let us prove de Moivre's Theorem by induction.*

a) What is the base case? Is there anything to prove or is the base case trivial?

b) Write down the inductive hypothesis for the inductive step.

c) For the inductive step, we wish to go from k to $k+1$. How do we do this? (Hint: We have a k^{th} power, which we want to make into a $(k+1)^{st}$ power.)

d) Using some trig identities, finish the inductive step and the proof.

2 Multiplication Revisited

Let us fix $r = 1$ and consider the function $f(\theta) = cos(\theta) + i\sin(\theta)$. De Moivre's Theorem tells us that

$$(f(\theta))^n = f(n\theta)$$

You can also maybe believe (or check it yourself - use those trig identities!) that

$$f(\alpha)f(\beta) = f(\alpha + \beta)$$

There is a function (actually, many functions) of real numbers that behaves a lot like f does. Namely, if we take a base b exponential for any real number $b > 1$, we see that

$$(b^x)^n = b^{nx} \text{ and } b^x b^y = b^{x+y}$$

For most purposes, it is convenient to choose our base to be e. (You'll just have to trust me on this for now, if you take calculus in school later this will make a lot more sense.) Therefore we can define

$$e^{i\theta} := \cos(\theta) + i\sin(\theta)$$

so that the following definition agrees with the one we gave last week.

Definition 2 *The polar form of a complex number is*

$$z = re^{i\theta}$$

where r is the modulus of z and θ is the argument of z.

Problem 2 *Let's write down formulas for adding complex numbers in both rectangular and polar forms.*

- Simplify $(a + bi) + (c + di)$ as much as you can.

- Simplify $re^{i\theta} + se^{i\phi}$ as much as you can.

- Which form is more convenient for addition?

Problem 3 *Now let's do multiplication.*

- Simplify $(a + bi) \times (c + di)$ as much as you can.

- Simplify $re^{i\theta} \times se^{i\phi}$ as much as you can.

- Which form is more convenient for multiplication?

Problem 4 *Recall from last week that we could not find a geometric interpretation of*

$$(1 + i) \times (2 + i)$$

Draw the picture again in the space below, and using the polar forms, give a geometric interpretation.

3 Dividing Complex Numbers

So far we have covered how to add, subtract, and multiply complex numbers. There is one more operation for real numbers we haven't discussed yet, which is division. Before we get to that, however, let us first define the complex conjugate.

Definition 3 *The complex conjugate of a complex number* $z = a + bi$ *(in rectangular form) is denoted* $\overline{z}$ *and is given by* $\overline{z} = a - bi$.

Problem 5 *We've defined conjugation in rectangular coordinates, but not in polar. So, what is* $\overline{re^{i\theta}}$*? (Hint: The way we've defined conjugation in rectangular makes it a reflection over the real line in the Argand plane. Think about the geometric implications of this.)*

Problem 6 *Prove that:*

- *For all real numbers* x, $\overline{x} = x$.
- *For all imaginary numbers* yi, $\overline{yi} = -yi$.

Problem 7 *Find the complex conjugates of each* z *below, in whichever form you would like.*

- $z = 3 + 4i$
- $z = 2e^{i\pi/6}$
- $z = \sqrt{2} + i\sqrt{2}$
- $z = 66e^{i\pi/27}$

Problem 8 *Recall that in rectangular form, complex numbers have a real and imaginary part. Prove that*

- $z + \overline{z} = 2Re(z)$
- $z - \overline{z} = 2iIm(z)$
- $z\overline{z} = |z|^2$

The above problem gives the (perhaps surprising) result that $z\overline{z}$ is always real (because in polar form it's equal to r^2, which is definitely real). Because we know how to divide things by real numbers, this gives us a way to finally define division:

$$\frac{w}{z} = \frac{w}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{w\overline{z}}{z\overline{z}}$$

and because we know how to multiply complex numbers and divide complex numbers by real numbers, this is good enough.

Problem 9 *Given $w = a + bi$ and $z = c + di$, provide a formula for dividing them in rectangular form.*

Problem 10 *Given $w = re^{i\theta}$ and $z = se^{i\phi}$, provide a formula for dividing them in polar form.*

Problem 11 *Which form is more convenient for dividing?*

Problem 12 *Give a geometric interpretation (drawing or description) of what it means to divide two complex numbers.*

4 Applications of Complex Numbers

As we have seen in the past two weeks, complex numbers are deeply related to angles in the plane. Naturally, when geometry problems have difficult angles, even if there's no complex numbers in the original problem, it can be ideal to put the picture into the complex plane. We illustrate the power of this technique with an example:

Problem 13 *(2017 ARML Individial #4) In triangle* ABC, $m\angle A = 90°$, $AC = 1$, *and* $AB = 5$. *Point* D *lies on ray* $\overrightarrow{AC}$ *such that* $m\angle DBC = 2m\angle CBA$. *Compute* AD.

a) Let us put this triangle into the complex plane. Where is the most convenient place to put the origin and the real/imaginary axes?

b) What is point C*, as a complex number? (Hint: Because we aligned a right triangle with real and imaginary axes, it will be most convenient to use the rectangular form.)*

c) The measure of angle $\angle DBA$ *equals the argument of a complex number* z. *Find* z *in rectangular form.*

d) Using z*, find point* D *as a complex number. Finish the problem from there.*

If you got to this page, congratulations! Here are some more difficult geometry problems for you to try, where all of them will involve complex numbers, even though none of them are given to you in the complex plane.

Problem 14 *(M. Lavrov) Show that given any quadrilateral, the midpoints of its sides form a parallelogram.*

Problem 15 *(A. Hildebrand) Show that*

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = \frac{4 - 2\cos(\theta)}{5 - 4\cos(\theta)}$$

Problem 16 *(MAA) Let $ABCD$ be a cyclic quadrilateral with $AB = 4, BC = 5, CD = 6,$ and $DA = 7$. Let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to line BD, and let B_1 and D_1 be the feet of the perpendiculars from B and D, respectively, to line AC. Find the perimeter of quadrilateral $A_1B_1C_1D_1$.*

Problem 17 *(E. Chen) Let ABC be a triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}, \overline{AB}$ by D, E respectively, and let O be the circumcenter of triangle ABC. Let P be a point such that $\Delta DPO \sim \Delta PEO$, and let X and Y be the midpoints of the major and minor arcs BC of the circumcircle of ΔABC. Find $PX \cdot PY$.*

Problem 18 *(T. Laurens) Prove that*

$$\tan^{-1}\left(\frac{x^2 + x}{x^3 + x^2 - 1}\right) - \tan^{-1}\left(\frac{1}{x^2 + x}\right) = \tan^{-1}\left(\frac{1}{x}\right) \text{ for all nonzero real numbers } x$$

Problem 19 *(T. Needham) On each side of triangle ABC put an equilateral triangle outside of ΔABC that shares that side with triangle ABC. Let P, Q, R be the new vertices created, and let X, Y, Z be the centroids of triangles ABP, BCQ, and CAR, respectively. Prove that XYZ is an equilateral triangle.*

Problem 20 *(E. Chen) Let P be a point in the plane of ΔABC, and γ a line passing through P. Let A', B', C' be the points where the reflections of the lines PA, PB, PC about γ intersect lines BC, AC, AB, respectively. Show that A', B', C' are collinear.*