Voting Theory

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1 Voting Systems

In this worksheet, we're going to take a mathematical look at elections. Elections with two candidates are pretty easy to understand. We have a set V of voters, we have our two candidates, and if every voter picks their favorite candidate, we need a system for deciding who wins the election. There's only one system that seems sensible - find out which candidate got the most votes, and they win. If there's an even number of voters, there could be a tie, but other than that, this is the best way to do it. But with more candidates, it gets more complicated (see any US Presidential Election with more than two major candidates). As several US jurisdictions are currently experimenting with different voting systems (Maine, New York City), let's take a look at the options. We'll think about what makes a voting system good or fair, and see if there's a best way to do many-candidate elections.

Definition 1. Let V be a set of voters, and C a set of candidates.

A *preference profile* is an ordering of the candidates.

A *voting system* for V and C is a function that takes as input a preference profile for each voter, and outputs a single preference profile, called the *outcome* or *result*.

Problem 1. 100 voters participated in an election with five candidates, A, B, C, D, E. They each ranked the five candidates, and the ballots came in as follows:

Number of voters	Preference Profile
31	$A > D > C > E > B$
20	$B > D > C > E > A$
19	$D > C > E > B > A$
16	$E > C > B > A > D$
14	$C > E > D > B > A$

Discuss possible voting systems that could be used for this election, and in each case, find out who would win. Can you find a fair voting system that makes each candidate win?

Problem 2. One criterion for a voting system is called *Pareto Efficiency* (PE): If there are two candidates, A and B, such that every voter thinks $A > B$, then the result has $A > B$.

Which of your voting systems satisfy this criterion? Explain why this might be a good criterion to have.

Problem 3. Another criterion for a voting system is called the *Independence of Irrelevant Alternatives* (IIA): Say there are two candidates, A and B, and in a particular election, $A > B$. If the voters change some of their preferences, but not the relative position of A and B, then the result still has $A > B$.

Which of your voting systems satisfy this criterion? Explain why this might be a good criterion to have.

Problem 4. What other criteria do you think a voting system needs to be fair? Which of our voting systems satisfy these criteria?

One more criterion: A system where the result only depends on one particular voter's preference profile is called *a dictatorship*. For very obvious reasons, we'd very much like to avoid those.

2 Ultrafilters

In the standard majority-rule voting system with two candidates, our voting system works in the following simple way: We split the set of voters into two disjoint sets, based on who they rank first. Then we compare these sets, see which one is larger, and go with their opinion. In this section, we'll try to see if we can generalize this description to work with more candidates.

In order to extend this to three candidates, we will similarly start by dividing the voter set into three disjoint subsets based on their favorite candidates - then we just need to decide which set will win. This leads us to a definition:

Definition 2. An *ultrafilter* on a set V is a set U of subsets of V such that any time V is partitioned into three disjoint parts, V_1, V_2, V_3, exactly one of V_1, V_2, and V_3 is in U.

Each ultrafilter U gives rise to a voting system on three candidates, as we can just divide the voters into three sets based on their favorite candidate, and exactly one set will be in U. We say that the candidate preferred by that set is the winner. (Technically we also need to decide the rest of the ranking, but we'll get to that later.) If we understand ultrafilters, that will help us understand the limitations of voting theory with three (or more) candidates. To get a bit of practice with this definition, let's look at the simplest kind of ultrafilter as an example:

Problem 5. Let V be a set, and $v \in V$ an element. Define U_v to be the set of all subsets of V that contain v. Check that U_v is an ultrafilter - we call any ultrafilter of this form *principal*.

Problem 6. Let U be an ultrafilter on V. Show that:

- $V \in U$
- $\emptyset \notin U$
- If V_1, V_2 are disjoint sets such that $V_1 \cup V_2 = V$, then exactly one of V_1, V_2 is in U.

This shows that we can also use U to choose a winner out of two candidates.

Problem 7. Let U be an ultrafilter on V.

- Let V_1, V_2 be disjoint subsets of V such that $V_1 \cup V_2 \in U$. Show that exactly one of V_1 and V_2 is in U.
- Show by induction on n that if $V_1, \ldots, V_n$ are disjoint subsets of V such that $V_1 \cup V_2 \cup \cdots \cup V_n = V$, then exactly one V_i is in U.
- Describe how you can use U to decide not only a winner, but a ranking of any (finite) number of candidates.

Problem 8. Let U be an ultrafilter on V, and let C be a finite set of candidates. Show that the voting system based on U satisfies PE and IIA.

Problem 9. Show that every ultrafilter on a finite set V is principal, and that the voting system based on that ultrafilter is a dictatorship.

2.1 Decisive Sets

Fix a voting system on V, with candidates C. A subset $X \subseteq V$ is called *decisive* if whenever every voter in X has the same preference profile, that preference profile is the outcome of the election.

Problem 10. Let C be a set of only two candidates, and let V be a set of an odd number of voters. If the voting system is standard majority rule, which subsets of V are decisive?

Problem 11. Let U be an ultrafilter on V, let C be a set of candidates, and use the voting system based on U. What are the decisive sets?

Problem 12. Assume that a voting system's decisive sets form an ultrafilter. Show that the voting system based on that ultrafilter is the original voting system - that is, there is only one voting system with that particular ultrafilter as its decisive sets.

Problem 13. Later on, we will show that the decisive sets in any voting system with at least three candidates that satisfies PE and IIA form an ultrafilter. Taking this as an assumption for now, prove *Arrow's Impossibility Theorem*: Any voting system with at least three candidates and finitely many voters satisfying PE and IIA is a dictatorship.

The rest of the worksheet is dedicated to showing that the decisive sets in any voting system with at least 3 candidates that satisfies PE and IIA form an ultrafilter. To do this, we'll first show that with only 2 or 3 voters, there is a dictator. Then we'll consider a larger voter base, and consider a partition of the voters into three pieces. We'll show that exactly one of them is decisive, by using our special case with only three voters.

To do that, we need a more specific definition. Let a, b be candidates. We say that a set X is *decisive for* (a, b) when any time every voter in X agrees that $a < b$, the outcome of the election also has $a < b$. Note that order matters here - it is not the same for X to be decisive for (b, a).

Problem 14. Show that with two voters in a system satisfying PE and IIA, if a, b are distinct candidates, there is a voter v such that $\{v\}$ is decisive for (a, b) or $\{v\}$ is decisive for (b, a).

Problem 15. Show that with two voters in a system satisfying PE and IIA, if a, b, c are distinct candidates, then if $\{v\}$ is decisive for (a, b), then it is also decisive for (a, c) and for (b, c). Use this to show that $\{v\}$ is decisive in general, and thus that this system is a dictatorship.

Problem 16. Consider a system with three voters $\{v_1, v_2, v_3\}$, satisfying PE and IIA, with at least three candidates. Define a new system on $\{v_1, v_2\}$ by assuming v_3 copies v_2's ballot, and finding the outcome of the three-voter system. Show that if $\{v_1\}$ is decisive in that two-voter system, then $\{v_1\}$ is decisive in the original three-voter system.

Problem 17. Keep considering a system with three voters satisfying PE and IIA, with at least three candidates. Show that there is a dictator.

Hint: Show by contradiction that there is some voter v such that if you group the other two voters together as in the last problem, $\{v\}$ is decisive.

Problem 18. Consider a voting system with at least three candidates satisfying PE and IIA. Show that the set of decisive candidates is an ultrafilter, finishing the proof of Arrow's Theorem.

Hint: You have to show that for every partition of the set of voters V into three disjoint pieces $V_1 \cup V_2 \cup V_3 = V$, that exactly one of V_1, V_2, and V_3 is decisive. To do this, first construct a voting system on three voters $\{v_1, v_2, v_3\}$, each of which represents one of our three sets, with everyone voting the same way. One of these voters must be a dictator. Show that the corresponding set in the partition is decisive in the original voting system.

3 Bonus: Ultrafilters In Infinite Combinatorics

So far, our discussion of ultrafilters has lacked interesting examples (principal ultrafilters aren't the most thrilling objects), and we haven't really seen why they earn the fantastic name of "ultrafilters". To explain the name, we will back up a bit and define *filters*, explain what makes ultrafiters so *ultra*, and then head to the world of infinite "voters" to see how the mathematical abstraction of nonprincipal ultrafilters, or "voting systems with PE and IIA that aren't dictatorships", can be useful in pure math.

3.1 Filters

Definition 3. Let V be a set (of metaphorical voters). Then a *filter* on V is a set F of subsets of V that satisfies these axioms:

- F is nonempty
- F is closed under intersections: if $A, B \in F$, then $A \cap B \in F$.
- F is closed upwards: if $A \in F$, and $A \subseteq B \subseteq V$, then $B \in F$.

We also say that a filter F is *proper* when F doesn't consist of *all* the subsets of V.

Let's do a few problems to get a feel for what this definition means:

Problem 19. Let F be a filter on V. Show that:

- $V \in F$
- F is proper if and only if $\emptyset \notin F$

We have some examples of proper filters lying around already, the ultrafilters:

Problem 20. Show that if U is an ultrafilter on V, then U is also a proper filter on V.

In order to move towards interesting filters and ultrafilters, we need to define what a *cofinite* subset of V is.

Definition 4. A subset $A \subseteq V$ is *cofinite* when $V \setminus A$ is finite.

Problem 21. Let F_F be the set of *cofinite* sets of V. Show that F_F is a filter (we call this the *Fréchet filter*, and that F_F is proper if and only if V is infinite.

Problem 22. Let U be an ultrafilter on V, and let F_F be the Fréchet filter on V. Show that U is nonprincipal if and only if $F_F \subseteq U$. (You can also use this to find another proof that all ultrafilters on finite sets are principal.)

We now need to introduce the theorem that powers our theory of nonprincipal ultrafilters:

Theorem 3.1. *If F is a proper filter on V, then there is an ultrafilter U extending F - that is, $F \subseteq U$.*

We won't prove this today - in fact, the proof uses the Axiom of Choice (or rather, a version of it called Zorn's Lemma). But we will use it!

Problem 23. Show that if V is infinite, there is a nonprincipal ultrafilter U on V.

3.2 Ramsey Theory

Let's turn our ultrafilters towards Ramsey Theory, where we find that large enough graphs must contain certain kinds of patterns. Specifically, imagine that you have a complete graph K_V on a set of vertices V - that is, every pair of distinct vertices is connected by an edge. You have a palette of k colors, and you color each *edge* with one of the k colors. The pattern we'd like to find is a *monochromatic set* of a certain size: that is, some set X of m vertices such that every edge between two vertices in X is the same color. Obviously we can't guarantee that we will have, say, a blue monochromatic subset of size 5, because the whole graph could be red. However, if the graph is large enough, we can find a monochromatic subset of size m for some color.

Theorem 3.2 (Finite Ramsey's Theorem)**.** *For any natural numbers k, m, there is some n such that if V is a set of at least n vertices, and we color K_V with k colors, there must be a monochromatic subset of size at least m.*

We're not going to prove this (though you may have seen a version of it in a previous Math Circle worksheet), but we will prove the infinite version. It's not immediate that one of these theorems implies the other, although there is a way to use ultrafilters to prove Finite Ramsey's Theorem from Infinite Ramsey's Theorem (perhaps a story for another day).

Theorem 3.3 (Infinite Ramsey's Theorem)**.** *For any natural number $k \geq 1$, if V is an infinite set of vertices, and we color the edges of K_V with k colors, there must be an infinite monochromatic subset.*

To prove this, we need a nonprincipal ultrafilter U on V. Thankfully, we know that such a thing exists, because V is infinite!

Problem 24. If $v \in V$, and c is one of our k colors, let A_v^c be the set of all vertices connected to v by an edge of color c. Show that for every vertex v, there is a unique color c such that $A_v^c \in U$.

Problem 25. For each color c, let V_c be the set of all vertices $v \in V$ such that A_v^c is in U. Show there is a unique c such that $V_c \in U$.

Problem 26. Let c be the color from Problem 25, such that $V_c \in U$.

Construct an infinite monochromatic subset of V_c of color c recursively.

Hint: Start by picking $v_0 \in V_c$. Then if you have $v_0, \dots, v_n$ that form a monochromatic subset of V_c of color c, use U to find a new element v_{n+1} of V_c that is connected to each of your existing vertices with an edge of color c.

3.3 Vertex Colorings

We can also use ultrafilters to prove things about colorings of the *vertices* of graphs. We say that a graph is *k-colorable* if there is a way to color the vertices of the graph with k colors such that no two vertices with the same color are connected by an edge. There are lots of theorems and algorithms for determining whether specific finite graphs are k-colorable, but what about infinite graphs? It turns out that with the power of ultrafilters, we can study k-colorability of infinite graphs by just understanding k-colorability of its finite subgraphs.

Theorem 3.4 (De Bruijn-Erdős)**.** *A graph is k-colorable if and only if all its finite subgraphs are.*

Problem 27. Show that if a graph G is k-colorable, and H is a subgraph of G, then H is k-colorable.

Now we will prove the other direction over the course of a few problems. Let G be a graph with vertices V and edges E. Let C be a palette of k colors. Assume that for each $X \in S_V$, the subgraph of G with vertices X is k-colorable, and pick a k-coloring for each such graph. (Technically we just used the axiom of choice, but hey, we're already throwing ultrafilters around.) We'd like to synthesize these colorings together into a coloring on the entire set of vertices V, and to do that, we're going to have the finite subgraphs vote on how the vertices should be colored! As each finite subgraph has an "opinion", to conduct this election, we'll need to construct an ultrafilter on the set S_V of all finite subsets of V.

Problem 28. Let F be the set of all subsets $Z \subseteq S_V$ such that for some finite $X \subseteq V$, all finite subsets of V that contain X are in Z. Show that F is a proper filter on S_V.

Problem 29. Let U be an ultrafilter extending F. Show that U is nonprincipal.

Now we will go back to colorings.

Problem 30. If $v \in V$, let A_v^c be the set of all subsets $X \in S_V$ such that X contains v, and in the k-coloring of X, v gets the color c. Show that for each $v \in V$, there is a unique $c \in C$ such that $A_v^c \in U$.

To construct our coloring, color each v with the unique color c such that $A_v^c \in U$.

Problem 31. Prove that this is actually a k-coloring - that is, check that no two adjacent vertices were given the same color.

4 References

Problem 1 is due to Alfonso Gracia-Saz.

For this approach to Arrow's Impossibility Theorem, my main source was this blog post by Sebastian Vasey. (Although be careful reading it, I think there may be an error of their proof of Lemma 16 - the voting system $S_{v,ab}$ doesn't satisfy unanimity (what we called PE), but they assume it does.)

The Ramsey Theory problems are adapted from these notes by David Galvin.

The proof of De Bruijn-Erdős follows this blog post by Chris Lambie-Hanson.