ORMC Olympiad Group
Week 8
Equations II

Osman Akar

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Theorem 1 (**Useful Identities**). *The following identities are must to know*

1. $x^2 - y^2 = (x - y)(x + y)$
2. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
3. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
4. $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$
5. *When n is odd*

   $$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots - xy^{n-2} + y^{n-1})$$

6. *Binomial Expansion*

   $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n-1} xy^{n-1} + y^n$$

Theorem 2 (**More Special Identities**). *The following identities are also useful to keep in mind*

1. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz)$

2. $x^2 + y^2 + z^2 - xy - xz - yz = \frac{1}{2}\left((x-y)^2 + (x-z)^2 + (y-z)^2\right)$

3. $x^4 + 4y^4 = x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 = (x^2+2y^2)^2 - (2xy)^2 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$

4. $(x+y)(x+z)(y+z)+xyz = (x+y+z)(xy+xz+yz) = \sum_{sym} x^2y + 3xyz$

5. $(x^2+y^2)(z^2+w^2) = (xz+yw)^2 + (xw-yz)^2$

Problems

1. **(TJNMO-FR 2014)** Find the number of integer pairs (x, y) so that

$$x^2 + 2xy = 4x + 3y^2$$

2. x, y, z are real numbers satisfying

$$\begin{aligned} x^2 + y^2 + 3z^2 &= 37 \\ x + y + 3z &= 11 \\ xy &= -3 \end{aligned}$$

   Find all possible values of z.

3. x, y, z are real numbers satisfying

$$\begin{aligned} x + y + z &= 13 \\ (x+y)^2 + z^2 &= 97 \\ xy + xz + yz &= 15 \end{aligned}$$

   Find all possible values that $x^3 + y^3 + z^3$ can take.

4. Real numbers x, y, z satisfies

$$x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 = 21, \qquad x + y + z = 6$$

   Find $x^2 + y^2 + z^2$.

5. Find the number of positive integer triples (a, b, c) so that

$$1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{239 - a - b - c}{abc}$$

6. **(TNMO-FR 2018)** Find $x^3 - 5x$ if x is irrational and both $x^2 - 2x$ and $x^3 - 5x$ are rational.
7. **(LAMC 2008)** Is it true that if all the sides of a triangle are less than 1 then the area is less than $\sqrt{3}/4$?
8. Find the number of solutions in positive reals to the following equation system
$$x^2 + 1 = 2y \qquad y^2 + 1 = 2z \qquad z^2 + 1 = 2x$$

New Problems

9. For real values x, y, z with $xy + xz + yz = 1000$, we know that $\frac{x^2+1000}{(x+y)^2} = \frac{1}{2020}$ and $\frac{y^2+1000}{(y+z)^2} = \frac{5}{4}$. What is
$$\frac{z^2 + 1000}{(x+z)^2}$$
10. We are given two real values a, b so that
    - $a - 2b = 3$
    - $a^2 + 2b^2 = 39$

    Find all possible values that $a + b$ can take.
11. **(TJNMO-FR 2016)** For a real number a, $21a + 2$ and $24a + 9$ are squares of two consecutive integers. The product of possible values of a can be written as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find mn.
12. **(AHSME 1999)** The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and
$$m^3 + n^3 + 99mn = 33^3$$
is equal to (A) 2 (B) 3 (C) 33 (D) 35 (E) 99
13. Let a, b, c be complex numbers satisfying the equations
$$\begin{aligned} a(b+1) + b(c+1) + c(a+1) &= 30 \\ a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc &= 189 \end{aligned}$$

Find the largest possible value of $a^2+b^2+c^2$.

14. a, b, c are real numbers satisfying the equations

$$\begin{aligned} a^3+b^2c+c^2b &= -83 \\ b^3+a^2c+ac^2 &= 29 \\ c^3+a^2b+ab^2 &= 349 \\ (a-1)^2+(b-1)^2+(c-1)^2 &= 52 \end{aligned}$$

Find the remainder of $ab+ac+bc$ when divided by 1000.

15. **(TJNMO-FR 2018)** a, b are nonnegative reals such that $\frac{a-7}{b}+\frac{b+7}{a}=2$. Find the sum all possible values of $a-b$.

16. Compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$$

17. **(AHSME 1999)** Let $x_1, x_2, \ldots, x_n$ be a sequence of integers such that

    (a) (i) $-1 \leq x_i \leq 2$ for $i = 1, 2, \ldots n$

    (b) (ii) $x_1+\cdots+x_n = 19$; and

    (c) (iii) $x_1^2+x_2^2+\cdots+x_n^2 = 99$.

    Let m and M be the minimal and maximal possible values of $x_1^3+\cdots+x_n^3$, respectively. Then find the fraction $\frac{M}{m}=$

18. Complex numbers x, y, z satisfies

$$x^2y+y^2z+z^2x = xy^2+yz^2+zx^2 = 21, \qquad x+y+z=6$$

Find the sum of all possible values of $x^2+y^2+z^2$