ORMC Olympiad Group
Week 7 Solutions

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Solutions

Solution 1

Note that

$$(1-a)(b) = (1-a)(a+1)(a^2+1)(a^4+1) = 1-a^8,$$

by repeatedly using difference of squares. Then, plugging in $a = -\frac{9}{10}$ into the LHS gives

$$\frac{19}{10}b = 1-a^8 \implies 19b + 10a^8 = \boxed{10}.$$

Solution 2

Let $2a - b = k$. We want to find all possible values of k. Substituting $b = 2a - k$ into the equation gives

$$\begin{aligned} 4a^2 + (2a-k+3)^2 &= 20 + 4(a+1)(2a-k+1) \\ \implies 4a^2 + 4a^2 + 4a(3-k) + (3-k)^2 &= 20 + 4(2a^2 + a(3-k) + (1-k)) \\ \implies (3-k)^2 &= 20 + 4(1-k) \\ \implies k^2 - 2k - 15 &= 0. \end{aligned}$$

Solving this quadratic then gives $k = \boxed{-3, 5}$.

Solution 3

Note that

$$(n-m)^2 + (k-n)^2 + (k-m)^2 = 2(m^2+n^2+k^2-mn-nk-mk) = 14.$$

But the only way to express 14 as the sum of squares of integers is as $1^2+2^2+3^2$. Thus, the differences $n-m, k-n, k-m$ are $1, 2, 3$ in some order. Clearly the largest difference is $k-m$, so $k-m=3$. Then, $n-m$ can be either 1 or 2, so the numbers are of the form $(m, m+1, m+3)$ or $(m, m+2, m+3)$. This means their sum is either $3m+4$ or $3m+5$, and $3m+4 = 2020$ gives $m = 672$ ($3m+5 = 2020$ does not give integer solutions). Thus, the answer is $(m, n, k) = \boxed{(672, 673, 675)}$.

Extension: How many solutions are there if the second condition is changed to $1 \leq m+n+k \leq 2022$?

Note that the solutions are of the form $(m, n, k) = (m, m+1, m+3)$ or $(m, m+2, m+3)$, so that $m+n+k = 3m+4$ or $3m+5$. But also, $m \geq 1$, so $m+n+k \geq 7$. Thus the number of solutions is the number of integers from 7 to 2022 which are 1 or 2 $\pmod 3$, which are

$$3\cdot 2+1, \quad 3\cdot 2+2, \quad \ldots, \quad 3\cdot 673+1, \quad 3\cdot 673+2.$$

Thus, there are $2(673-2+1) = \boxed{1344}$ solutions.

Solution 4

We would like to add the equations and be able to manipulate it such that there is a sum of squares which equals 0, which would then force the constraints for a solution. This is not possible if the equations are simply added; so, we multiply the second equation by some constant factor k and then add them. This gives

$$2x^2 - 3y + ky^2 - 4kx = -\frac{17}{2} + 7k$$
$$\implies 2(x-k)^2 + k\left(y - \frac{3}{2k}\right)^2 = 2k^2 + k\left(\frac{3}{2k}\right)^2 - \frac{17}{2} + 7k,$$

after completing the square. Then, we want the RHS to be 0, which would then give $x = k$ and $y = \frac{3}{2k}$. If the RHS is 0, then

$$2k^2 + \frac{9}{4k} - \frac{17}{2} + 7k = 0 \implies 8k^3 + 28k^2 - 34k + 9 = 0.$$

Then we see that $k = \frac{1}{2}$ is a root of this polynomial. So, $x + y = k + \frac{3}{2k} = \boxed{\frac{7}{2}}$.

Solution 5

Completing the square gives

$$(x + 42)^2 + 244 = y^2 \implies (y + x + 42)(y - x - 42) = 244.$$

Note that $244 = 2^2 \cdot 61$ and a solution in integers would lead to the two terms on the LHS having the same parity. Thus it must be $122 \cdot 2$, so

$$y + x + 42 = 122, \quad y - x - 42 = 2 \implies y = 62, x = 18 \implies x + y = \boxed{080}.$$

Solution 6

The equation rearranges to $x^2 + x(2y - 4) - 3y^2 = 0$. Then by quadratic formula,

$$x = \frac{4 - 2y \pm \sqrt{(2y-4)^2 - 4(-3y)^2}}{2} = 2 - y \pm \sqrt{4y^2 - 4y + 4}.$$

Then this means $4y^2 - 4y + 4$ is a perfect square. Since y is an integer, we can remove the factor of 4, so we have $y^2 - y + 1 = m^2$ for some integer m. This rearranges to $y^2 - y + (1 - m^2) = 0$; by quadratic formula,

$$y = \frac{1 \pm \sqrt{1 - 4(1 - m^2)}}{2}.$$

Thus, $1 - 4(1 - m^2) = (2k + 1)^2$ for some integer k. This rearranges to $(2m - 2k - 1)(2m + 2k + 1) = 3$. Since 3 can only factor as $\pm 3 \cdot \pm 1$, and checking these possibilities gives $m = \pm 1$. Thus, $m^2 = 1$, so we have $y^2 - y + 1 = 1 \implies y = 0, 1$. Then plugging this back in gives:

$$y = 0 \implies x = 2 \pm 2, \quad y = 1 \implies 1 \pm 2.$$

Thus there are $\boxed{4}$ solutions in integers to the equation.

Solution 7

We have $x^2 + y^2 = 37 - 3z^2$, but also, $x + y = 11 - 3z$ and $xy = -3$. Since $x^2 + y^2 = (x + y) - 2xy$, substituting in the expressions gives

$$37 - 3z^2 = (11 - 3z)^2 - 2(-3) \implies 2z^2 - 11z + 15 = 0,$$

after simplifying. This factors as $(2z - 5)(z - 3) = 0$, so $z = \boxed{\frac{5}{2}, 3}$.

Solution 8

From the first equation, $x+y=13-z$, and plugging this into the second equation gives $(13-z)^2+z^2=97$. The solutions to this quadratic are $z=4,9$. Then, note that from combining the first and third equations,

$$x^2+y^2+z^2=13^2-2(15)=139.$$

But the second equation is $x^2+2xy+y^2+z^2=97$, so substituting the above gives $xy=-21$. Then, note that

$$\begin{aligned} x^3+y^3+z^3 &= 3xyz+(x+y+z)(x^2+y^2+z^2-xy-xz-yz) \\ &= 3xyz+13(139-15) \\ &= 1612-63z. \end{aligned}$$

Since $z=4,9$, plugging these in gives $x^3+y^3+z^3=\boxed{1045, 1360}$.

Solution 9

We can rearrange the first given equation to

$$x^2y+y^2z+z^2x-xy^2-yz^2-zx^2=0,$$

and then the left side factors:

$$(x-y)(x-z)(y-z)=0.$$

WLOG suppose $x=y$ holds. Then, $x^3+x^2z+xz^2=21$ and $2x+z=6$, so subsituting $z=6-2x$ into the first equation and simplifying gives

$$3x^3-18x^2+36x-21=0.$$

Then, it is easy to see that $x=1$ is a solution to this equation, so then $x=y=1$ and $z=4$. Thus $x^2+y^2+z^2=\boxed{18}$.

Solution 10

Multiplying the equation by abc and rearranging gives

$$abc+ab+bc+ca+a+b+c=239.$$

Adding 1 to each side of the equation allows the left side to be factored, giving

$$(a+1)(b+1)(c+1)=240=2^4\cdot 3\cdot 5.$$

First we will count the number of ordered triples of positive integers multiplying to 240, and then subtract the cases where some of these equal 1 (because if $a+1=1$ then $a=0$ which is not allowed). To do this, we consider each prime factor separately.

The power of 2 is 4, so these 4 2's must be distributed to the 3 numbers in some way. The number of ways to do this, by a stars and bars approach, is $\binom{6}{2}=15$. Similarly, there are $\binom{3}{2}=3$ ways to assign each of the 3 and the 5. So, the total number of ways is $15\cdot 3\cdot 3=135$.

Now we count the number of such ordered triples where at least one number is 1. If one of them is 1, there are 3 ways to choose which one and then $5\cdot 2\cdot 2=20$ divisors of 240, so there are $3\cdot 20$ ways here. But this double counts the triples with two numbers equal to 1; these are just the permutations of $(1, 1, 240)$. Thus, there are $3(20) - 3 = 57$ triples where at least one number is 1.

Thus the final answer is $135 - 57 = \boxed{78}$.

Solution 11

Let $x^2 - 2x = r \in \mathbb{Q}$. Then, note that

$$x^3 - 5x = x(x^2 - 2x) + 2(x^2 - 2x) - x = xr + 2r - x.$$

So, $x(r-1) + 2r \in \mathbb{Q}$. Since $2r \in \mathbb{Q}$, this means $x(r-1) \in \mathbb{Q}$. But x is irrational and $r - 1$ is rational, so the only way this is possible is if $r - 1 = 0$, i.e. $r = 1$. Then, $x^3 - 5x = x(r-1) + 2r = x(0) + 2 = \boxed{2}$.

Solution 12

The answer is $\boxed{\text{yes}}$. Consider a triangle with side lengths a, b, c, and suppose $a, b, c \leq 1$. By Heron's formula, the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)}.$$

Then, note that by AM-GM and the given $a, b, c \leq 1 \implies a + b + c \leq 3$,

$$(a+b-c)(a-b+c)(-a+b+c) \leq \left(\frac{a+b+c}{3}\right)^3 \leq \left(\frac{1+1+1}{3}\right)^3 = 1.$$

Thus,

$$\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)} \leq \sqrt{\frac{(a+b+c)(1)}{16}} \leq \sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4},$$

as desired.

Solution 13

Note that by AM-GM, $2y = x^2 + 1 \geq 2x$. Similarly, $2z = y^2 + 1 \geq 2y$ and $2x = z^2 + 1 \geq 2z$. So,

$$2x \geq 2z \geq 2y \geq 2x.$$

Thus, equality must hold, so there is only $\boxed{1}$ solution, $x = y = z = 1$.