# Dynamical Systems

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## 1 Preliminaries

First, let's have a brief refresher on convergence of sequences. We won't be needing all the formal stuff today, so we'll use informal definitions.

**Definition 1.1.** A sequence $x_1, x_2, x_3, \dots \in \mathbb{R}$ is said to *converge to* $x \in \mathbb{R}$ if the numbers in the sequence become closer and closer to $x$ and not to any other number.

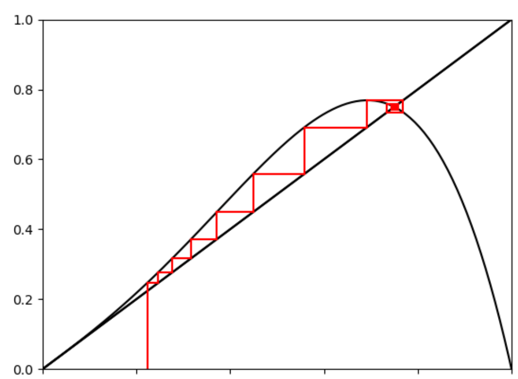
**Problem 1.** Do the following sequences converge to anything? If so, what do they converge to?

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$
2. $1, 2, 3, 4, 5, \dots$
3. $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1 \dots$
4. $3, 3.1, 3.14, 3.141, 3.1415, \dots$
5. $1, 4, 1.9, 3.1, 1.99, 3.01, 1.999, 3.001, \dots$

**Problem 2.** Can a sequence converge to more than one value?

Let $f$ be a real-valued function such that if $0 \leq x \leq 1$, then $0 \leq f(x) \leq 1$. To draw a *cobweb plot* of $f$, first plot the functions $y = x$ and $y = f(x)$. Pick a starting value $x_0 \in [0, 1]$. Start by plotting a line from $(x_0, 0)$ to $(x_0, f(x_0))$, then draw a line from $(x_0, f(x_0))$ to $(f(x_0), f(x_0))$. Keep connecting dots, alternating between vertical lines $(x, x)$ to $(x, f(x))$ and horizontal lines $(x, f(x))$ to $(f(x), f(x))$.

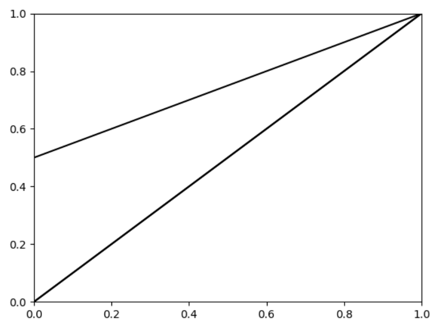
Here's an example for $f(x) = x(1 - x)(4x^2 + x + 1)$, with $x_0 = 0.223$:



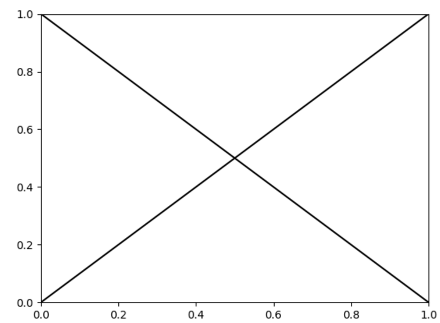

*with revisions by Glenn Sun

**Problem 3.** Let $f(x) = x(1-x)(4x^2+x+1)$, $x_0 = 0.223$. Using the cobweb plot, does the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converge? If so, what can you say about its limit?

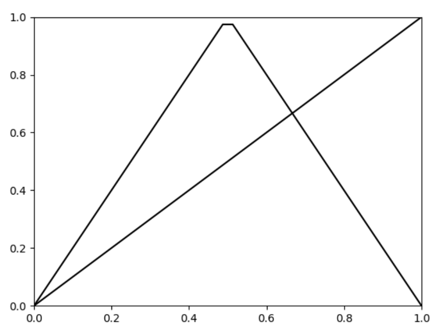
**Problem 4.** Fill out the following cobweb plots with starting point $x_0 = \frac{1}{5}$:



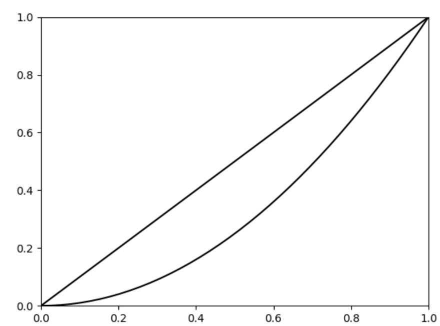

(a) $f(x) = \frac{x+1}{2}$




(b) $f(x) = 1 - x$




(c) $f(x) = 1 - 2|x - 0.5|$




(d) $f(x) = x^2$

**Definition 1.2.** A fixed point of a function is a point $x \in \mathbb{R}$ satisfying $f(x) = x$. We say that a fixed point $x$ of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *stable* if there is some interval $(a, b)$ containing $x$ such that if $x_0 \in (a, b)$, then the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converges to $x$. Otherwise it is called *unstable*.

**Problem 5.** In the four examples in the previous problem, where are the fixed points, if there are any? Are they stable or unstable? (You might want to draw a few cobweb plots with starting points closer to the fixed points to figure this out.)

## 1.1 Optional Bonus Problems

**Problem 6.** Let $f(x) = ax + b$ be a linear function. For what values of $a, b$ does $f$ have a fixed point, and when does $f$ have a stable fixed point?

A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous if whenever $x_0, x_1, x_2, \ldots$ converges to $x$, the sequence $f(x_0), f(x_1), f(x_2), \ldots$ converges to $f(x)$.

**Problem 7.** By definition, every stable fixed point is the limit of some convergent sequence $x_0, f(x_0), f(f(x_0)), \ldots$. Is the converse always true? That is, if $x_0, f(x_0), f(f(x_0))$ converges to $x$, must $x$ be a stable fixed point, or at least a fixed point? Give a condition on $f$ for when $x$ must be a fixed point, give an example of $f$ and $x_0$ where $x$ is not a fixed point, and give an example of $f$ and $x_0$ where $x$ is a fixed point but not stable.

## 2 Logistic Maps

In this section, we study some functions called the *logistic maps*, which pop up in population modelling, and also have interesting mathematical properties. (Note: These are different (though related to) the *logistic curve*, which also pops up in population modelling problems and in calculus classes. If you've heard of that, you can think of these logistic maps as a discrete-time version of that continuous-time population model, but this discrete version has much more personality.) Let's model the size of a population. Suppose there is a species of rabbit that has a fixed generation length. If the rabbits have an infinite supply of food, then after each generation, each rabbit is replaced with $r$ rabbits of the next generation.

**Problem 8.**

1. Given this infinite supply of food, if we start with $x$ rabbits at generation 0, how many rabbits do we have at generation $t$?
2. Under what circumstances will the number of rabbits approach a constant population?

Now assume that the rabbit's reproductive rate depends on the amount of available food, and that the amount of available food depends on the number of rabbits. Assume that their environment has a carrying capacity, a limit to number of rabbits that the food can support. Let's measure the population not as a natural number, counting the rabbits, but as a real number, $x$, which is the fraction of the carrying capacity, so that $x = 0$ indicates 0 rabbits, but $x = 1$ indicates that the population is the carrying capacity. Now assume that with each successive generation, each rabbit is replaced with $r(1-x)$ children, so that as the number of rabbits increases to the carrying capacity, and the amount of available food decreases to 0, the reproductive rate shrinks down from $r$ to 0.

**Definition 2.1.** If there are $x$ rabbits at generation $t$, then there will be $rx(1-x)$ rabbits at generation $t+1$. We call this function the *logistic map with parameter* $r$, and will use the notation

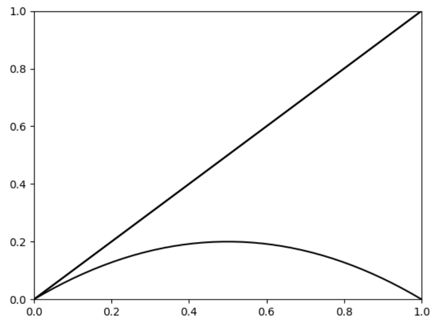
$$\boxed{x_{t+1} = f_r(x_t) = rx_t(1-x_t)}$$

**Problem 9.**

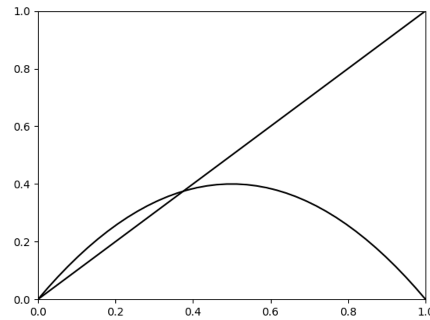
1. Describe the trends in population if $0 \leq r < 1$.
2. Explain what happens in our model if we let $x$ at generation $t$ be greater than 1.
3. What population at generation 0 maximizes the population at generation 1?
4. We know that our model isn't necessarily predictive if we ever have $x$ outside the interval $[0, 1]$. Do all values of $r$ guarantee that if we start with $x \in [0, 1]$, it stays in that interval forever? If not, what values do guarantee this?

**Problem 10.** What are the fixed points of $f_r$? Are they valid populations?

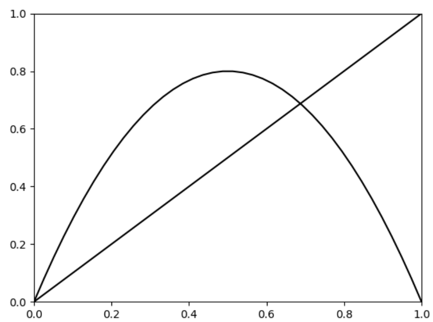
**Problem 11.** Draw a cobweb diagram for $f_r$ at $r = 0.8, 1.6, 3.2, 3.5$, with one or more different starting values of $0 < x < 1$. In each of the diagrams, say whether or not the fixed points are stable.



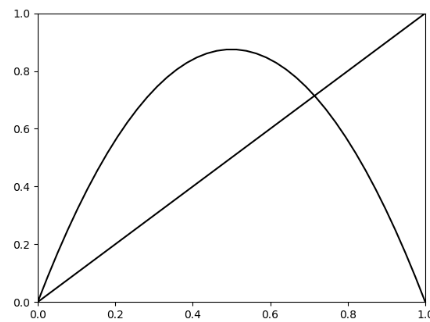

(a) $r = 0.8$




(b) $r = 1.6$




(c) $r = 3.2$




(d) $r = 3.5$

**Problem 12.** Make a copy of the following Google Sheet: `https://bit.ly/ormc-logistic`. (If you're in-person, feel free to use your phone or ask an instructor to pull it up for you.) The Google Sheet calculates the same sequence $x_{t+1} = f_r(x_t)$ that you found in the cobweb diagram. By adjusting the value of $r$ within the range you found in Problem 9.4, for what values of $r$ does the population converge to a fixed point? For each such $r$, which fixed point does it converge to? Make sure your answer agrees with the previous examples in your cobweb plots.

The proof of this takes some work, so we put it in the bonus section.

**Problem 13.** For values of $r$ between 3 and 4, when is there some regularity in what happens? When does the system become chaotic?

## 2.1 Optional Bonus Problems

The problems here build up the following theorem.

**Theorem 2.1.** For $0 \le r \le 1$, 0 is the unique fixed point and is stable. For $1 \le r \le 3$, 0 is unstable and $1 - \frac{1}{r}$ is stable. For $3 < r < 4$, no fixed point is stable.

**Problem 14.** Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant polynomial. Recall that the finitely many zeroes of $f$ split $\mathbb{R}$ into intervals on which $f$ is positive and intervals on which $f$ is negative. Show that for a fixed point $x_0$ to be stable, $f(x) - x$ must be positive immediately to the left of $x_0$ or negative immediately to the right of $x_0$.

**Problem 15.** Prove that 0 is a stable fixed point for $0 \leq r \leq 1$, and that it is unstable for $1 < r \leq 4$.

We will find the following result about limits useful:

**Problem 16.** Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers, let $0 \leq r < 1$, and let $a$ be a number. Show that if for all $n$, $|a_{n+1} - a| < r|a_n - a|$, then $a_0, a_1, a_2, \ldots$ converges to $a$.

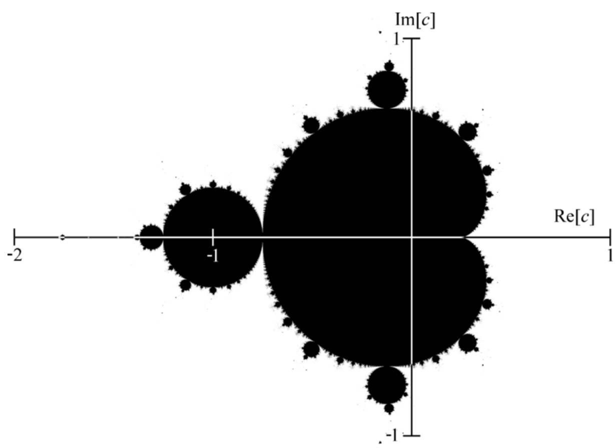
**Problem 17.** Let $f(x) = ax^2 + bx + c$ be a quadratic, and let $x_0$ be a fixed point of $f(x)$. Show that if $|2ax_0 + b| < 1$, then $x_0$ is stable. (Hint: use the previous problem.)

**Problem 18.** Let $f(x) = ax^2 + bx + c$ be a quadratic, and let $x_0$ be a fixed point of $f(x)$. Show that if $|2ax_0 + b| > 1$, then $x_0$ is unstable.

**Problem 19.** Finish the proof of Theorem 2.1.

## 3 The Mandelbrot Set

The Mandelbrot set is a subset of $\mathbb{C}$ that you might have heard of: it's a fractal, meaning there is infinite detail as you keep zooming in.




(Image source: Wikipedia.) We will now investigate the Mandelbrot set and its connection to the logistic map. First, we recall some definitions about complex numbers.

**Definition 3.1.**

- Recall that the complex numbers $\mathbb{C}$ can all be expressed as $a+bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
- If $a, b \in \mathbb{R}$, let $\overline{a + bi} = a - bi$. We call this the *complex conjugate* of $a + bi$.
- If $a, b \in \mathbb{R}$, let $|a + bi| = \sqrt{a^2 + b^2}$. We call this the *magnitude* of $a + bi$.

**Definition 3.2.** Let $z_0, z_1, z_2, \ldots$ be a sequence of complex numbers. Say that this sequence is *bounded* if and only if there is some real number $M$ such that for all $n \in \mathbb{N}$, $|z_n| \leq M$.

**Problem 20.** Determine which of the following sequences of complex numbers are bounded:

1. $0, 1, 2, 3, \ldots$
2. $1, \frac{1}{2}, \frac{1}{3}, \ldots$
3. $z_n = \cos(n) + i \sin(n)$
4. $\frac{1}{1}, \frac{1}{1} + \frac{1}{2}, \frac{1}{1} + \frac{1}{2} + \frac{1}{3}, \ldots$

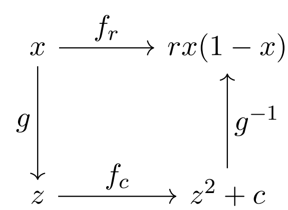
**Definition 3.3.** The *Mandelbrot set* is the set of all complex numbers $c$ such that the sequence $z_0, z_1, \ldots$ is bounded, where $z_0 = 0$ and $z_{n+1} = f_c(z_n) = z_n^2 + c$.

**Problem 21.** Determine whether the following points are in the Mandelbrot set, that is, if the sequence defined above is bounded for the following choices of $c$: $0, 1, -1, i, -i, \frac{1}{4}$.

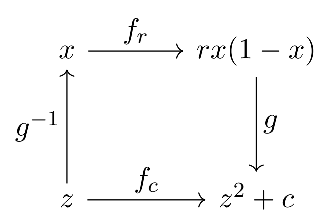
Recall that $f_r(x) = rx(1 - x)$ is the logistic map. The Mandelbrot set has a deep connection with logistic maps. Actually, they are pretty much the same thing: it's possible to implement the logistic map using the Mandelbrot map.

Recall that if $g(x)$ is any bijective (one-to-one and onto) function, $g^{-1}$ is the inverse of $g$, satisfying $g^{-1}(g(x)) = x$ for all $x$. In other words, $g^{-1}$ undoes $g$.

We can compute the logistic map $f_r(x)$ using the Mandelbrot map as follows: Let $z = g(x)$ where $g(x) = ax + b$ for some particular $a$ and $b$. Then, take Mandelbrot map $f_c(z)$ for some particular $c$. Finally, transform $f_c(z)$ back using $g^{-1}$, that is, $f_r(x) = g^{-1}(f_c(z))$. We will determine what $a, b, c$ should be in the next question.




Similarly, we can also compute the Mandelbrot map $f_c(z)$ using the logistic map. The process is exactly the same, except that we start with $g^{-1}$.




**Problem 22.** We will investigate the connection described above.

1. Let $r$ be given. Show that $g(x) = ax + b$ works for some real numbers $a$ and $b$. That is, solve for $a, b, c$ in the following equation in terms of $r$:

$$g(x)^2 + c = g(rx(1 - x)).$$

2. Practice using this correspondence. Let $x_0 = \frac{1}{4}$ and $r = 2$. Compute $x_1$, $x_2$, and the corresponding $z_0$, $z_1$, and $z_2$, using whatever method you like. Then, compute the same values a different way using the correspondence.

3. Explain why a sequence given by the logistic map $x_0, f_r(x_0), f_r(f_r(x_0)), \ldots$ is bounded if and only if the corresponding sequence given by the Mandelbrot map $z_0, f_c(z_0), f_c(f_c(z_0)), \ldots$ is also bounded.

We will now investigate the shape of the Mandelbrot set using this correspondence. Look back a few pages for the picture.

**Problem 23.** We first describe the intersection of the Mandelbrot set with the real number line.

1. Recall that behavior given by the logistic map $x_n = f_r(x_n)$, $x_0 \in [0, 1]$ produces behavior bounded in $[0, 1]$ if and only if $r \in [0, 4]$. Are there more $r$ for which the behavior of the logistic map is bounded, although not necessarily in $[0, 1]$? Feel free to investigate again with the Google Sheet.

2. Using the correspondence from the previous question, describe the intersection of the Mandelbrot set with the real number line. Compare your answer with the picture from a few pages ago.

Next, notice that the main part of the Mandelbrot set consists of one big heart-shaped region (called a cardioid), surrounded by what looks like circles. Does each region have some kind of special property?

**Problem 24.** Recall that logistic map has qualitatively different behavior in certain intervals:

- For $r \in [0, 1]$, sequences converge to 0.
- For $r \in [1, 3]$, sequences converge to a non-zero fixed point.
- For $r$ slightly above 3, sequences converge to periodic behavior.
- For $r$ slightly below 4, sequences exhibit chaotic behavior.

Now, do the following problems.

1. Using the correspondence from Problem 22, graph $c$ as a function of $r$. (On paper or on Desmos.) Then look at the picture of the Mandelbrot set. Trace $c$ from $r = -2$ to 4 in the picture. For what $r$ are there "key" transitions between regions? Make a prediction for what $c$ belong in the large cardioid.

2. Recall that from Problem 11, $r = 3.2$ gives periodic behavior with period 2, and $r = 3.5$ gives periodic behavior with period 4. What $c$ do these correspond to? By looking at the picture of the Mandelbrot set again, can you make some more conjectures?

**Problem 25.** We will find a parametric equation for the boundary of the cardioid region.

1. Stable fixed points can be defined for functions $f : \mathbb{C} \to \mathbb{C}$, in essentially the same way they were defined over the reals: a point satisfying $f(z) = z$ such that for all $z_0$ near $z$, iterating $z_0, f(z_0), f(f(z_0)), \ldots$ converges to $z$.

   Show that if $r$ is complex, then $f_r(z) = rz(1-z)$ defined on the complex numbers has a stable fixed point if $|r| < 1$. (For real numbers, this means $r \in [-1, 1]$: remember that $r \in [1, 3]$ also generates stable fixed points, we'll treat this case later.)

2. Any complex $r$ with $|r| = 1$ can be expressed as $\cos\theta + i\sin\theta$ for some real $\theta$. Find a parametric equation which consists of all $c \in \mathbb{C}$ corresponding to $|r| = 1$. Graph this on Desmos. What do you see?

3. You might be surprised by what you saw, since we only used $|r| \leq 1$, ignoring $r$ such as 2 for which there are nonzero stable fixed points. Why do we not need to consider those other $r$?

To understand the other parts of the Mandelbrot set, it helps to introduce some new words. We've been mentioning periodic behavior without defining it for some time now, so we give the definition here.

**Definition 3.4.** Let $n > 0$ be a natural number and $f : \mathbb{C} \to \mathbb{C}$ be a function. Then we say that $z \in \mathbb{C}$ is a *periodic point* of $f$ when $f^n(z) = \underbrace{f(f(\dots(f}_{n \text{ times}}(z))\dots) = z$. We say that *$z$ has period $n$* if $n$ is the least positive number such that this is true.

For instance, under the map $f(z) = -z$, every point is periodic, and every point has period 2 except for 0 which has period 1. A point of period 1 is the same as a fixed point.

**Problem 26.**

1. Suppose $f_c(z) = z^2 + c$ has a periodic point $z$ of period 2, but not period 1. Write an equation that $z$ satisfies if and only if this is true.

2. What are the two periodic points of period 2 when $c = -1$? When $c = -\frac{7}{4}$?

3. For $c = -1$ or $-\frac{7}{4}$, suppose you start iterating $z_0, f_c(z_0), \dots$ very close to a periodic point, but not exactly at the periodic point. Do the values approach the periodic points or go away from it? Feel free to use a calculator.

Because of this stability requirement, which we would need calculus to analyze properly, deriving the equation for the circle to the left of the main cardioid is actually significantly harder than the cardioid, and beyond the scope of this worksheet. However, its shape is easily guessable.

**Problem 27.**

1. Looking at the plot of the Mandelbrot set, guess an equation that describes the boundary of the circle to the left of the main cardioid.

2. Using this, what is the largest value of $r$ for which $x_0, f_r(x_0), \dots$ becomes a cycle with period 2?

## 3.1 Optional Bonus Problems

**Problem 28.** We will show that the Mandelbrot set is contained in the disk of radius 2 centered at the origin.

1. Let $c, z \in \mathbb{C}$ be such that $|z| > 2$. Then show that $|f_c(z)| - |c| \geq 2(|z| - |c|)$.

2. Let $z_{n+1} = f_c(z_n)$. Show that if for some $n$, $|z_n| > 2$ and $|z_n| > |c|$, then the sequence $z_n, z_{n+1}, z_{n+2}, \dots$ is unbounded.

3. Conclude that if for some $n$, $|z_n| > 2$ and $|z_n| > |c|$, then $c$ is not in the Mandelbrot set.

4. Finally, show that if $|c| > 2$, then $c$ is not in the Mandelbrot set, and conclude that the Mandelbrot set is contained in the disk of radius 2 centered at the origin.

## 4 Video

To find out more about these topics, Youtube has many great videos about the logistic map, the Mandelbrot set, and the connections between the two.

Here is one of my favorites, on the logistic map, which touches on the Mandelbrot set: `https://www.youtube.com/watch?v=ovJcsL7vyrk`