ORMC Olympiad Group
Week 6 Solutions

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Solutions

Solution 1

(a) We want to find the largest k such that $x + 2y = k$ and $x^2 + y^2 = 4$ has a solution. Plugging in $x = k - 2y$ gives

$$(k-2y)^2 + y^2 = 4 \implies 5y^2 - 4ky + (k^2 - 4) = 0.$$

In order for this to have a solution, the discriminant must be nonnegative. The discriminant is $(-4k)^2 - 4(k^2-4)(5) = 80 - 4k^2$, so

$$80 - 4k^2 \geq 0 \implies k^2 \leq 20 \implies k \in [-2\sqrt{5}, 2\sqrt{5}].$$

Thus, the largest possible value of k is $\boxed{2\sqrt{5}}$.

(b) By Cauchy-Schwarz,

$$(x^2+y^2)(1+4) \geq (x+2y)^2.$$

Thus,

$$x + 2y \leq \sqrt{(x^2+y^2)(1+4)} = \sqrt{(4)(5)} = \boxed{2\sqrt{5}}.$$

(c) Notice that $x^2 + y^2 = 4$ describes a circle with radius 2 centered at the origin, and $x + 2y = k$ is the equation of a line with slope $-\frac{1}{2}$ and y-intercept $\frac{k}{2}$. The system of equations has a solution if these two intersect. Note that if we keep increasing k, at some point the line will be tangent to the circle, after which any higher value of k will lead to a line that does not intersect the circle.

So, we solve for k given that the line is tangent to the circle. This can be done with similar triangles to find $k = \boxed{2\sqrt{5}}$.

Solution 2

(a) We apply AM-GM on each term of the LHS:

$$a + b \geq 2\sqrt{ab}, \qquad a + c \geq 2\sqrt{ac}, \qquad b + c \geq 2\sqrt{bc}.$$

Multiplying these inequalities gives the desired $(a+b)(a+c)(b+c) \geq 8abc$.

(b) By AM-GM, we have $(x_i + 1) \geq 2\sqrt{x_i}$, for each $1 \leq i \leq n$. So, multiplying these gives

$$\prod_{i=1}^{n}(x_i+1) \geq \prod_{i=1}^{n} 2\sqrt{x_i} = 2^n \sqrt{\prod_{i=1}^{n} x_i} = 2^n,$$

as desired.

Solution 3

It is given that $x^2 + y^2 = 1$. So, by Cauchy-Schwarz,

$$(x^2+y^2)(9+16) \geq (3x+4y)^2 \implies 25 \geq (3x+4y)^2.$$

Thus, $-5 \leq 3x + 4y \leq 5$. So the maximum and minimum values are $\boxed{5, -5}$.

Solution 4

By Cauchy Schwarz,

$$(x^2+y^2+z^2)(3^2+4^2+12^2) \geq (3x+4y+12z)^2 \implies 3x+4y+12z \leq \sqrt{3^2+4^2+12^2} = \boxed{13}.$$

Equality occurs when $\frac{x^2}{3^2} = \frac{y^2}{4^2} = \frac{z^2}{12^2}$. Then combining $x^2+y^2+z^2=1$ and $x^2:y^2:z^2 = 3^2:4^2:12^2$, (and since $3^2+4^2+12^2=13^2$, we have

$$x^2 = \frac{3^2}{13^2}, \quad y^2 = \frac{4^2}{13^2}, \quad z^2 = \frac{12^2}{13^2} \implies (x,y,z) = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right).$$

Solution 5

By Cauchy Schwarz, we have

$$(x^2+y^2)(4+1) \geq (2x+y)^2 \iff 5B \geq A^2.$$

From checking the answer choices, we see that only answer choice $\boxed{\textbf{(C)}}$ does not satisfy this inequality.

Solution 6

Suppose the side lengths of the box are a, b, c, and let the lengths of the diagonals of the faces be x, y, z. Then, we have that $x^2 = a^2+b^2$, $y^2 = a^2+c^2$, and $z^2 = b^2+c^2$ (in some order). Then, note that

$$a^2 = \frac{x^2+y^2-z^2}{2} \implies \frac{x^2+y^2-z^2}{2} \geq 0.$$

(since it equals the square of some number). So, we have that

$$x^2+y^2-z^2 \geq 0, \qquad y^2+z^2-x^2 \geq 0, \qquad z^2+x^2-y^2 \geq 0.$$

So, we must find the triple (x,y,z) from the answer choices which does not satisfy all of the above. The answer is then $\boxed{\textbf{(B)}}$, since $4^2+5^2-7^2 < 0$.

Solution 7

We present three solutions.

(i) We rewrite each term in the form $\frac{a^2}{ab+bc}$ and then apply Cauchy Schwarz:

$$\left(\frac{a^2}{ab+bc} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb}\right)((ab+bc)+(bc+ba)+(ca+cb)) \geq (a+b+c)^2.$$

So,

$$\frac{a^2}{ab+bc} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

Then, note that since $a^2+b^2+c^2 \geq ab+bc+ca$ (proved last week with AM-GM), we have that $(a+b+c)^2 \geq 3(ab+bc+ca)$. Thus,

$$\frac{a^2}{ab+bc} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2},$$

as desired.

(ii) Let $x = a + b$, $y = b + c$, and $z = c + a$. Then, note that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{(x+z-y)/2}{y} + \frac{(x+y-z)/2}{z} + \frac{(y+z-x)/2}{x}$$
$$= \frac{1}{2}\left(\frac{x}{y} + \frac{z}{y} - 1 + \frac{x}{z} + \frac{y}{z} - 1 + \frac{y}{x} + \frac{z}{x} - 1\right).$$

But then by AM-GM, we have that

$$\frac{x}{y} + \frac{z}{y} + \frac{x}{z} + \frac{y}{z} + \frac{y}{x} + \frac{z}{x} \geq 6\sqrt[6]{1} = 6,$$

so

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}(6-3) = \frac{3}{2}.$$

(iii) We rewrite each term in the form $\dfrac{a+b+c}{b+c} - 1$, so that we now want to prove

$$\frac{a+b+c}{b+c} - 1 + \frac{a+b+c}{c+a} - 1 + \frac{a+b+c}{a+b} - 1 \geq \frac{3}{2} \iff (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{9}{2}.$$

Equivalently, we must prove

$$2(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 9.$$

Now, rewrite $2(a+b+c)$ as $(b+c) + (c+a) + (a+b)$ and apply Cauchy Schwarz:

$$((b+c) + (c+a) + (a+b)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq (1+1+1)^2 = 9,$$

as desired.

Solution 8

By AM-GM, we have

$$x^2y = x^3 + y^4 = \frac{x^3}{3} + \frac{x^3}{3} + \frac{x^3}{3} + y^4 \geq 4\sqrt[4]{\frac{x^9y^4}{27}}.$$

Cancelling out the y gives

$$x^2 \geq 4\sqrt[4]{\frac{x^9}{27}} \implies \frac{27}{256} \geq x$$

after simplification. Thus $A = \dfrac{27}{256}$. Similarly, by AM-GM again, we have

$$x^2y = x^3 + y^4 = \frac{x^3}{2} + \frac{x^3}{2} + y^4 \geq 3\sqrt[3]{\frac{x^6y^4}{4}}.$$

Cancelling out the x^2 gives

$$y \geq 3\sqrt[3]{\frac{y^4}{4}} \implies \frac{4}{27} \geq y$$

after simplification. Thus $B = \dfrac{4}{27}$. So,

$$\frac{A}{B} = \frac{27/256}{4/27} = \boxed{\frac{729}{1024}}.$$

Solution 9

Consider a triangle XYZ with $\angle XYZ = 120^\circ$, $XY = a$, and $YZ = c$. Then, let W be on the internal angle bisector of $\angle XYZ$, such that $\angle XYW = \angle ZYW = 60^\circ$, and $YW = b$.

Then, note that by Law of Cosines,

$$XW = \sqrt{a^2 - ab + b^2}, \quad ZW = \sqrt{b^2 - bc + c^2}, \quad XZ = \sqrt{a^2 + ac + c^2}.$$

Thus, the desired inequality is equivalent to $XW + ZW \geq XZ$, which is true by triangle inequality on ΔXWZ. Equality occurs when $W \in XZ$.

Solution 10

Let $P = BC \cap MN$, $Q = AC \cap MN$. Let $\frac{CM}{CN} = x$. Then, note that

$$\frac{MP}{NP} = \frac{[BMC]}{[BNC]} = \frac{BM \cdot CM \sin \angle BMC}{BN \cdot CN \sin \angle BNC} = \frac{3x}{4}.$$

Then, since $MP + NP = 1$,

$$\frac{MP}{1 - MP} = \frac{3x}{4} \implies MP = \frac{3x}{3x+4}.$$

Similarly,

$$\frac{MQ}{NQ} = \frac{[AMC]}{[ANC]} = \frac{AM \cdot CM \sin \angle AMC}{AN \cdot CN \sin \angle ANC} = x,$$

so $MQ = \frac{x}{x+1}$. Then, since $MQ - MP = d$, we have

$$\frac{x}{x+1} - \frac{3x}{3x+4} = d \implies 3dx^2 + (7d-1)x + 4d = 0.$$

Since there must be a solution for x, the discriminant must be nonnegative; this gives that

$$d^2 - 14d + 1 \geq 0.$$

The roots of this are at $d = 7 \pm 4\sqrt{3}$ by the quadratic formula; the discriminant is negative between these roots, and nonnegative otherwise; so, it is nonnegative for $d \in (-\infty, 7 - 4\sqrt{3}] \cup [7 + 4\sqrt{3}, \infty)$. But also, note that $d \in (0, 1)$ since $MN = 1$; thus, the values of d satisfying all constraints of the problem are $d \in (0, 7 - 4\sqrt{3}]$, so the maximum value of d is $7 - 4\sqrt{3}$. Then, $r + s + t = \boxed{014}$.

Solution 11

The answer is $\boxed{\text{yes}}$. Consider a triangle with side lengths a, b, c, and suppose $a, b, c \leq 1$. By Heron's formula, the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)}.$$

Then, note that by AM-GM and the given $a, b, c \leq 1 \implies a + b + c \leq 3$,

$$(a+b-c)(a-b+c)(-a+b+c) \leq \left(\frac{a+b+c}{3}\right)^3 \leq \left(\frac{1+1+1}{3}\right)^3 = 1.$$

Thus,

$$\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)} \leq \sqrt{\frac{(a+b+c)(1)}{16}} \leq \sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{4},$$

as desired.