ORMC Olympiad Group
Week 7
Equations

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Theorem 1 (Useful Identities). *The following identities are must to know*

1. $x^2 - y^2 = (x-y)(x+y)$
2. $x^3 - y^3 = (x-y)(x^2+xy+y^2)$
3. $x^3 + y^3 = (x+y)(x^2-xy+y^2)$
4. $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$
5. *When n is odd*

   $$x^n + y^n = (x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\cdots-xy^{n-2}+y^{n-1})$$

6. *Binomial Expansion*

   $$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+y^n$$

Theorem 2 (More Special Identities). *The following identities are also useful to keep in mind*

1. $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz)$

2. $x^2+y^2+z^2-xy-xz-yz=\frac{1}{2}\left((x-y)^2+(x-z)^2+(y-z)^2\right)$

3. $x^4+4y^4=x^4+4x^2y^2+4y^4-4x^2y^2=(x^2+2y^2)^2-(2xy)^2=(x^2-2xy+2y^2)(x^2+2xy+2y^2)$

4. $(x+y)(x+z)(y+z)+xyz=(x+y+z)(xy+xz+yz)=\sum_{sym}x^2y+3xyz$

5. $(x^2+y^2)(z^2+w^2)=(xz+yw)^2+(xw-yz)^2$

Problems

1. **(TJNMO-FR 2008)** Given $a=-\frac{9}{10}$ and $b=(a+1)(a^2+1)(a^4+1)$. Calculate $19b+10a^8$.

2. Two positive real values a,b satisfies the equality

$$4a^2+(b+3)^2=20+4(a+1)(b+1)$$

   Find all possible values that $2a-b$ can take.

3. Find all positive integer triples (m,n,k) that satisfies
   - $m\le n\le k$
   - $m+n+k=2020$
   - $m^2+n^2+k^2=mn+mk+nk+7$

4. **(TJNMO-FR 2009)** Real numbers x and y satisfies the equations

$$2x^2-3y=-8.5$$

$$y^2-4x=7$$

   Find $x+y$.

5. **(AIME 2008I)** There exist unique positive integers x and y that satisfy the equation

$$x^2+84x+2008=y^2$$

   Find $x+y$.

6. **(TJNMO-FR 2014)** Find the number of integer pairs (x, y) so that

$$x^2 + 2xy = 4x + 3y^2$$

7. x, y, z are real numbers satisfying

$$\begin{aligned} x^2 + y^2 + 3z^2 &= 37 \\ x + y + 3z &= 11 \\ xy &= -3 \end{aligned}$$

   Find all possible values of z.

8. x, y, z are real numbers satisfying

$$\begin{aligned} x + y + z &= 13 \\ (x + y)^2 + z^2 &= 97 \\ xy + xz + yz &= 15 \end{aligned}$$

   Find all possible values that $x^3 + y^3 + z^3$ can take.

9. Real numbers x, y, z satisfies

$$x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 = 21, \qquad x + y + z = 6$$

   Find $x^2 + y^2 + z^2$.

10. Find the number of positive integer triples (a, b, c) so that

$$1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{239 - a - b - c}{abc}$$

11. **(TNMO-FR 2018)** Find $x^3 - 5x$ if x is irrational and both $x^2 - 2x$ and $x^3 - 5x$ are rational.

12. **(LAMC 2008)** Is it true that if all the sides of a triangle are less than 1 then the area is less than $\sqrt{3}/4$?

13. Find the number of solutions in positive reals to the following equation system

$$x^2 + 1 = 2y \qquad y^2 + 1 = 2z \qquad z^2 + 1 = 2x$$