Mathematical Induction, Peano Axioms, and Properties of Addition of Non-Negative Integers

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Contents

1 Introduction

The goal of this lecture is to prove that addition of non-negative integers is *associative*,

$$(l + m) + n = l + (m + n), \tag{1.1}$$

and *commutative*,

$$m + n = n + m, \tag{1.2}$$

for any l, m, and n in $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$. Nearly every grown-up is familiar with the properties. Very few can explain why they hold.

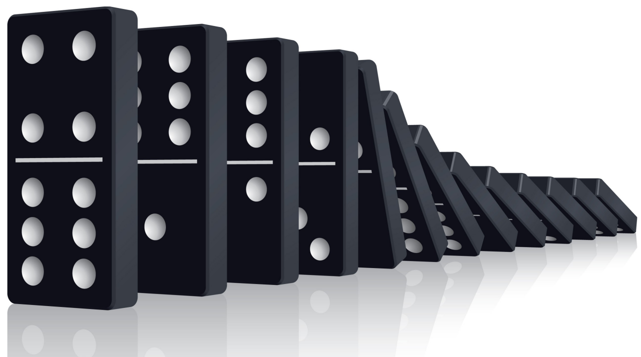
It is a recent tendency to call the numbers $\{0, 1, 2, 3, \ldots\}$ *natural.* Traditionally natural numbers are defined as the elements of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. The difference in notations is insignificant for Mathematics, but it is very important from the historical standpoint. People have mastered natural numbers as early as they have started counting, tens, if not hundreds, of thousands years ago. The ingenious idea to reserve a special symbol for *nothing* has occurred to humanity much later, somewhere in the middle of the first millennium AD. It requires a way higher degree of intellectual sophistication than just labelling the counters 1, 2, etc.

In order to prove associativity and commutativity of addition in the simplest possible case of non-negative integers, we will have to teach a blank mind what these numbers are and to develop their properties from scratch. Imagine building an Artificial Intelligence (AI). The AI has no knowledge of either the outside world or of its internal, cognitive one – you are only about to construct it. When a child learns to count, she/he already understands intuitively what 1 is. The child also knows quite a few facts regarding simple counting. There is no need to prove to her/him that $1 + 1 = 2$. She/he knows that one toy and another toy is two toys from everyday practice. The AI under construction has no idea of counters. So far, it has no idea at all! To make its thinking efficient, we need to hard-wire into its electronic brain as few simple rules of counting as possible. The rest, the entire arithmetic, including associativity and commutativity of addition, should follow from the rules.

2 Mathematical induction

Suppose that we have an infinite list of related mathematical statements S_n where n are either natural numbers $1, 2, 3 \ldots$ or non-negative integers $0, 1, 2, 3 \ldots$ The first statement is called the *base case.* Suppose that S_1 is true. If we establish the *inductive step* by proving that S_n implies S_{n+1}, then we prove the validity of the statements S_n for any and all n. Indeed, $S_1 \Rightarrow S_2$, $S_2 \Rightarrow S_3$, $S_3 \Rightarrow S_4$, and so forth.

An example of mathematical induction is the *domino effect.* Imagine that we have an infinite set of dominoes lined up at equal distances along a straight line. Imagine further that the distance between the dominoes is short enough for a falling domino to force the fall of the next one.



Let us prove an infinite list of related statements

$$S_n = the\ nth\ domino\ falls$$

by induction.

The base case: the first domino falls. We prove it by inspection. Give the first domino a nudge and see what happens. If it falls, this proves the base case. If it doesn't, then the domino effect may not occur.

The inductive hypothesis: assume that S_n is true, the nth domino falls.

The inductive step: thinking S_n is true, prove that S_{n+1} is true as well. Proof – the falling nth domino forces the fall of the $n+1$ one.

This way, the fall of the first domino forces the fall of the second, the fall of the second forces the fall of the third, and so forth.

The following famous formula

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{2.3}$$

was anecdotally discovered by Gauss at the age of three. (Outside of mathematical texts, the Greek letter Σ is pronounced as *sigma*. In mathematical texts, it means and reads a *sum*.) Here is Gauss's proof. Let us write down the sum twice, reversing the order of the summands the second time.

$$\begin{array}{llllllllllll}\Sigma = & 1 & + & 2 & + & \ldots & + & (n-2) & + & (n-1) & + & n \\ \Sigma = & n & + & (n-1) & + & \ldots & + & 3 & + & 2 & + & 1\end{array}$$

Adding the sums term-by-term produces the following.

$$2\Sigma = (n+1) + (n+1) + \ldots + (n+1) + (n+1) = n(n+1)$$

Dividing both sides by two proves (2.3).

To practice mathematical induction, let us use it to give a different proof to formula (2.3).

The base case: $n = 1$. The equality $1 = 1(1+1)/2$ is checked by inspection.

The inductive hypothesis: assume that formula (2.3) is true.

The inductive step: based on the assumption, prove that $1+2+3+\ldots+n+(n+1) = (n+1)(n+2)/2$. Note that the right-hand side of the latter formulas equals the right-hand side of formula (2.3) with n replaced by $n+1$.

$$1+2+\ldots+n+(n+1) = \underline{1+2+\ldots+n} + (n+1) =$$

$$\frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}$$

The sequence $p_1 = a$, $p_2 = a+q$, $p_3 = a+2q$, ... $p_n = a+(n-1)q$ is called an *arithmetic sequence* or an *arithmetic progression*. The sequence $1, 2, 3, \ldots$ we have summed up above is an example. Here is one more: $2, 5, 8, 11, 14, \ldots$

Question 2.1 *What is a and what is q in this case?*

The following formula

$$\sum_{i=1}^{n} p_i = a + (a+q) + \ldots + (a+(n-1)q) = na + q\frac{n(n-1)}{2} \qquad (2.4)$$

is a minor generalization of (2.3).

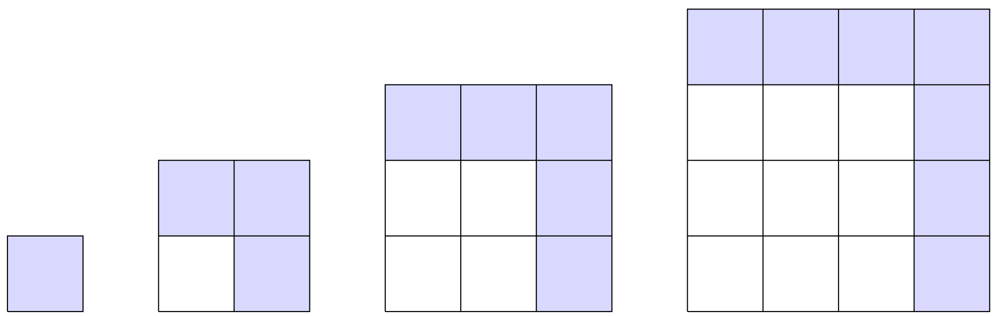
Problem 2.1 *Derive (2.4) from (2.3).*

Problem 2.2 *Use mathematical induction to prove (2.4) directly.*

Problem 2.3 *Use formula (2.4) for the sum of an arithmetic sequence to prove the following identity.*

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2 \tag{2.5}$$

Problem 2.4 *Consider the pictures below to prove formula (2.5) using geometry rather than algebra.*



Problem 2.5 *Prove (2.5) using mathematical induction.*

Problem 2.6 *Use mathematical induction to prove the following.*

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{2.6}$$

Problem 2.7 *Experiment with sums of cubes of the form* $1^3 + 2^3 + 3^3 + \ldots + n^3$ *for various natural* n*. Try to find a formula similar to (2.3) and (2.6). Then use mathematical induction to prove it.*

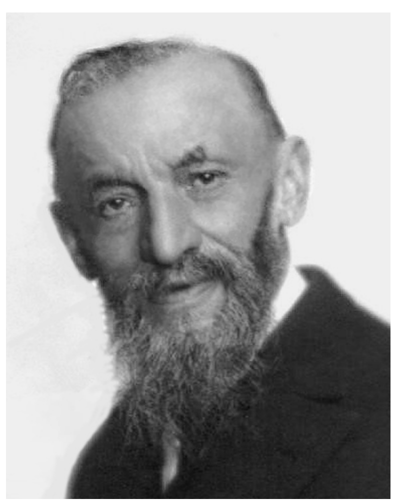
The following remarkable formula holds for the sum of the fourth powers of the first n natural numbers.

$$\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \tag{2.7}$$

Problem 2.8 *Prove (2.7).*

3 Peano axioms and properties of addition

We will call *Peano axioms* five simple rules below that will allow us to start teaching AI to count. The name honors an Italian mathematician, Giuseppe Peano, the inventor of the axiomatic approach to arithmetic.



Giuseppe Peano, 1858 – 1932

Our rules differ from the original axioms suggested by Peano, but our way of thinking will closely follow his approach.

P1: there exists a non-empty set of *non-negative integers*. Since the set is not empty, it has at least one element. An element of the set is called 0.

So far, 0 is just a label without any meaning except for the fact that it is an element of the set $\mathbb{Z}_{\geq 0}$ we are about to construct.

P2: There exists a *unary operation*, called *succession* and denoted as S, that takes a non-negative integer as an input and produces a non-negative integer as an output.

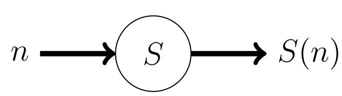
Let us use the operation of succession to construct the set of non-negative integers.

$$S(0) = 1, \quad S(1) = 2, \quad S(2) = 3 \quad S(3) = 4, \ldots$$

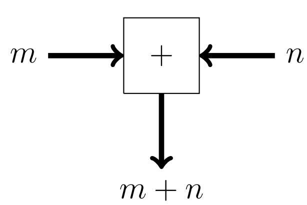
The symbols 1, 2, 3, etc. are just labels. Instead of the Arabic numbers, you can use the Roman ones, or anything else.

$$S(0) = I, \quad S(I) = II, \quad S(II) = III \quad S(III) = IV, \ldots$$

What we get is not just a set, but an ordered set, or a *list*. The zeroth element of the list is the original number, 0. The first element of the list is $S(0)$. The second element of the list is $S(S(0))$. The n-th element of the list is produced by applying the operation of succession to zero n times. The elements of the list are constructed inductively.

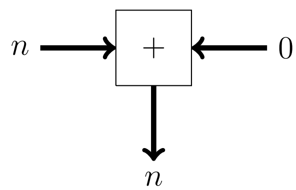


P3: There exists a *binary operation*, called *addition* and denoted as $+$, that takes two non-negative integers as inputs, one on the left and another on the right, and produces a non-negative integer as an output.



Note that $m + n$ is nothing more than the output label.

P4: If the right input of the addition operation is 0, then the output equals the left input.



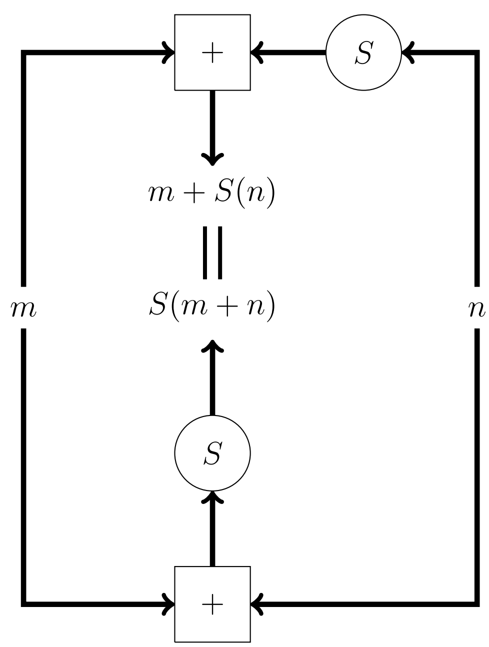
In other words, it is postulated that 0 is *right-neutral* with respect to addition,

$$n + 0 = n \tag{3.8}$$

for any non-negative integer n.

The last rule postulates how the operations of succession and addition interact with each other. Unlike the previous rules, this one may not seem natural for the first look.

P5: $m + S(n) = S(m + n)$. The fifth axiom requires the following two sequences of operations to produce the same result.



Theorem 3.1 $S(n) = n + 1$

All the proofs of this lecture are very formal. They are carried out using the Claim-Reason charts. Humans most often do not need such a level of formalism, but the AI we are building does. This is the only way of thinking available to the machine!

Proof of Theorem 3.1

Claim	**Reason**
$n + 0 = n$	**P4**
$S(n) = S(n + 0)$	$n + 0 = n$
$S(n + 0) = n + S(0)$	**P5**
$S(0) = 1$	The definition of the symbol 1.
$S(n) = n + 1$	All of the above.

Corollary 1 $1 + 1 = 2$

Proof of Corollary 1

Claim	**Reason**
$2 = S(1)$	The definitions of the symbols 1 and 2.
$S(1) = 1 + 1$	Theorem 3.1.
$2 = 1 + 1$	All of the above.

Our AI has started learning to count!

Problem 3.1 *Use a Claim-Reason chart to prove that* $2 + 2 = 4$.

Associativity is the following property of addition.

Theorem 3.2 $(l + m) + n = l + (m + n)$

Proof of Theorem 3.2: by induction on the third summand, n.

The base case: for any two natural numbers l and m,
$(l + m) + 0 = l + m$ and $l + (m + 0) = l + m$. Reason: **P4**. Therefore, $(l + m) + 0 = l + (m + 0)$.

To make the step of induction, we need to show that the inductive hypothesis, $(l+m)+n = l+(m+n)$, implies

$$(l+m)+S(n) = l+(m+S(n))$$

for any natural numbers l and m.

Problem 3.2 *Finish the proof of Theorem 3.2 by providing reasons for the claims that complete the inductive step.*

Claim	**Reason**
$(l+m)+S(n) = S((l+m)+n)$	
$(l+m)+n = l+(m+n)$	
$S((l+m)+n) = S(l+(m+n))$	
$S(l+(m+n)) = l+S(m+n)$	
$l+S(m+n) = l+(m+S(n))$	
$(l+m)+S(n) = l+(m+S(n))$	

Axiom **P4** postulates that 0 is right-neutral with respect to addition, $n+0=n$. The following lemma shows that zero is left-neutral with respect to addition as well.

Lemma 1 $0+n=n$

Problem 3.3 *Prove Lemma 1 by induction on n.*

Corollary 2 *The operation of adding zero is commutative, $0+n=n+0$ for any non-negative integer n.*

The following lemma establishes another special case of commutativity of addition. We will need it as a tool to prove Theorem 3.3.

Lemma 2 $1+n=n+1$

Proof of Lemma 2: by induction on n.

The base case: $n = 0$. Then $1 + 0 = 0 + 1$. Reason: Corollary 2.

Problem 3.4 *Finish the proof of Lemma 2 by providing reasons for the claims that complete the inductive step.*

Claim	**Reason**
$1 + S(n) = S(1 + n)$	
$1 + n = n + 1$	
$S(1 + n) = S(n + 1)$	
$n + 1 = (n + 1) + 0$	
$S(n + 1) = S((n + 1) + 0)$	
$S((n + 1) + 0) = (n + 1) + S(0)$	
$n + 1 = S(n)$	
$S(0) = 1$	
$(n + 1) + S(0) = S(n) + 1$	
$1 + S(n) = S(n) + 1$	

Finally, we are ready to prove the main theorem of this lecture.

Theorem 3.3 *For any two non-negative integers m and n, $m + n = n + m$.*

Proof of Theorem 3.3: by induction on n.

The base case: $n = 0$. Proven in Corollary 2.

Problem 3.5 *Finish the proof of Theorem 3.3 by providing reasons for the claims that complete the inductive step.*

Claim	**Reason**
$m + S(n) = S(m + n)$	
$m + n = n + m$	
$S(m + n) = S(n + m)$	
$S(n + m) = (n + m) + 1$	
$(n + m) + 1 = n + (m + 1)$	
$m + 1 = 1 + m$	
$n + (m + 1) = n + (1 + m)$	
$n + (1 + m) = (n + 1) + m$	
$n + 1 = S(n)$	
$(n + 1) + m = S(n) + m$	
$m + S(n) = S(n) + m$	

4 Philosophical note

Zero stands for *nothing* while one can be *anything different from nothing*. But as soon as you have one, you have two. As soon as you have two, you have three, and so on. Isn't this what somebody quite knowledgeable has tried to explain to our distant ancestors? "The earth was without form and void, and darkness was over the face of the deep. And the Spirit of God was hovering over the waters..." Before this world was born, there was nothing and there was something else. From the point of view of a mathematician, the Book of Genesis begins with the Peano axioms.