Dynamical Systems

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1 Preliminaries

First, let's have a brief refresher on convergence of sequences. We won't be needing all the formal stuff today, so we'll use informal definitions.

Definition 1.1. A sequence $x_1, x_2, x_3, \dots \in \mathbb{R}$ is said to *converge to* $x \in \mathbb{R}$ if the numbers in the sequence become closer and closer to x and not to any other number.

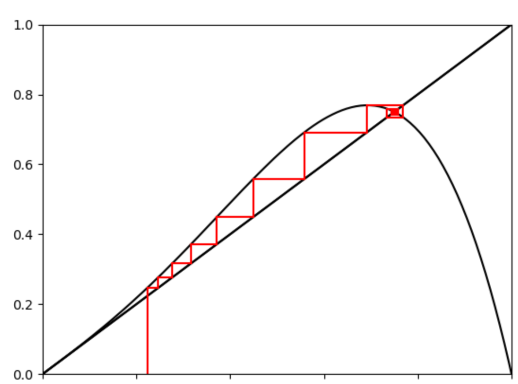
Problem 1. Do the following sequences converge to anything? If so, what do they converge to?

1. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$
2. $1, 2, 3, 4, 5, \dots$
3. $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1 \dots$
4. $3, 3.1, 3.14, 3.141, 3.1415, \dots$
5. $1, 4, 1.9, 3.1, 1.99, 3.01, 1.999, 3.001, \dots$

Problem 2. Can a sequence converge to more than one value?

Let f be a real-valued function such that if $0 \le x \le 1$, then $0 \le f(x) \le 1$. To draw a *cobweb plot* of f, first plot the functions $y = x$ and $y = f(x)$. Pick a starting value $x_0 \in [0, 1]$. Start by plotting a line from $(x_0, 0)$ to $(x_0, f(x_0))$, then draw a line from $(x_0, f(x_0))$ to $(f(x_0), f(x_0))$. Keep connecting dots, alternating between vertical lines (x, x) to $(x, f(x))$ and horizontal lines $(x, f(x))$ to $(f(x), f(x))$.

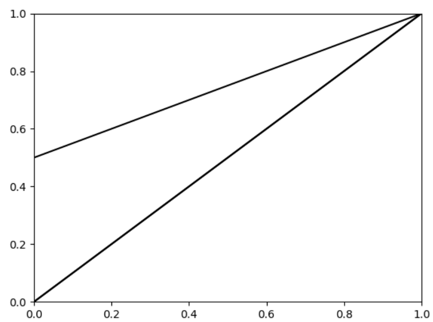
Here's an example for $f(x) = x(1 - x)(4x^2 + x + 1)$, with $x_0 = 0.223$:



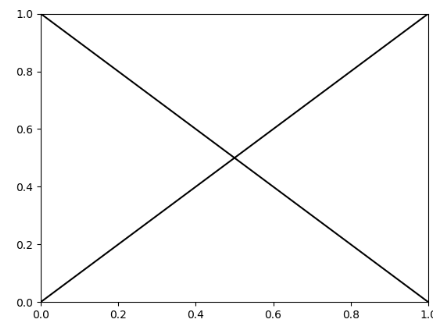
*with revisions by Glenn Sun

Problem 3. Let $f(x) = x(1-x)(4x^2+x+1)$, $x_0 = 0.223$. Using the cobweb plot, does the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converge? If so, what can you say about its limit?

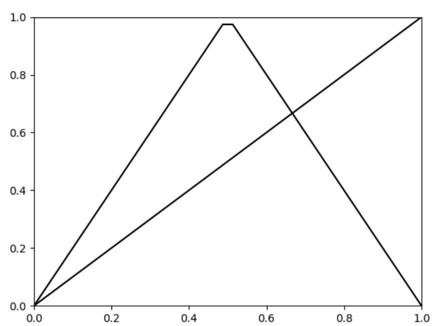
Problem 4. Fill out the following cobweb plots with starting point $x_0 = \frac{1}{5}$:



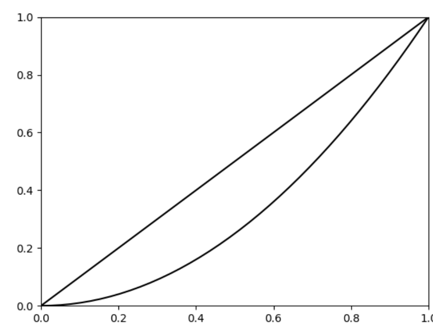
(a) $f(x) = \frac{x+1}{2}$



(b) $f(x) = 1 - x$



(c) $f(x) = 1 - 2|x - 0.5|$



(d) $f(x) = x^2$

Definition 1.2. A fixed point of a function is a point $x \in \mathbb{R}$ satisfying $f(x) = x$. We say that a fixed point x of a continuous function $f : \mathbb{R} \to \mathbb{R}$ is *stable* if there is some interval (a, b) containing x such that if $x_0 \in (a, b)$, then the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converges to x. Otherwise it is called *unstable.*

Problem 5. In the four examples in the previous problem, where are the fixed points, if there are any? Are they stable or unstable? (You might want to draw a few cobweb plots with starting points closer to the fixed points to figure this out.)

1.1 Optional Bonus Problems

Problem 6. Let $f(x) = ax + b$ be a linear function. For what values of a, b does f have a fixed point, and when does f have a stable fixed point?

A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous if whenever $x_0, x_1, x_2, \ldots$ converges to x, the sequence $f(x_0), f(x_1), f(x_2), \ldots$ converges to $f(x)$.

Problem 7. By definition, every stable fixed point is the limit of some convergent sequence $x_0, f(x_0), f(f(x_0)), \ldots$. Is the converse always true? That is, if $x_0, f(x_0), f(f(x_0))$ converges to x, must x be a stable fixed point, or at least a fixed point? Give a condition on f for when x must be a fixed point, give an example of f and x_0 where x is not a fixed point, and give an example of f and x_0 where x is a fixed point but not stable.

2 Logistic Maps

In this section, we study some functions called the *logistic map*s, which pop up in population modelling, and also have interesting mathematical properties. (Note: These are different (though related to) the *logistic curve*, which also pops up in population modelling problems and in calculus classes. If you've heard of that, you can think of these logistic maps as a discrete-time version of that continuous-time population model, but this discrete version has much more personality.) Let's model the size of a population. Suppose there is a species of rabbit that has a fixed generation length. If the rabbits have an infinite supply of food, then after each generation, each rabbit is replaced with r rabbits of the next generation.

Problem 8.

1. Given this infinite supply of food, if we start with x rabbits at generation 0, how many rabbits do we have at generation t?
2. Under what circumstances will the number of rabbits approach a constant population?

Now assume that the rabbit's reproductive rate depends on the amount of available food, and that the amount of available food depends on the number of rabbits. Assume that their environment has a carrying capacity, a limit to number of rabbits that the food can support. Let's measure the population not as a natural number, counting the rabbits, but as a real number, x, which is the fraction of the carrying capacity, so that $x = 0$ indicates 0 rabbits, but $x = 1$ indicates that the population is the carrying capacity. Now assume that with each successive generation, each rabbit is replaced with $r(1-x)$ children, so that as the number of rabbits increases to the carrying capacity, and the amount of available food decreases to 0, the reproductive rate shrinks down from r to 0.

Definition 2.1. If there are x rabbits at generation t, then there will be $rx(1-x)$ rabbits at generation $t+1$. We call this function the *logistic map with parameter* r, and will use the notation

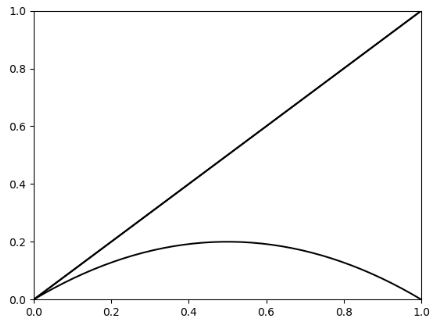
$$\boxed{x_{t+1} = f_r(x_t) = rx_t(1-x_t)}$$

Problem 9.

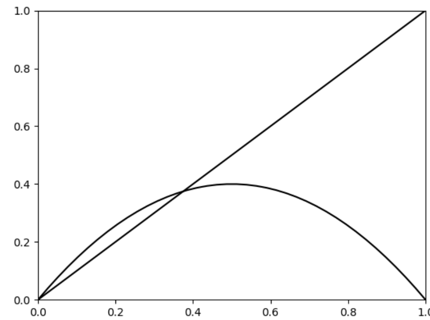
1. Describe the trends in population if $0 \leq r < 1$.
2. Explain what happens in our model if we let x at generation t be greater than 1.
3. What population at generation 0 maximizes the population at generation 1?
4. We know that our model isn't necessarily predictive if we ever have x outside the interval $[0, 1]$. Do all values of r guarantee that if we start with $x \in [0, 1]$, it stays in that interval forever? If not, what values do guarantee this?

Problem 10. What are the fixed points of f_r? Are they valid populations?

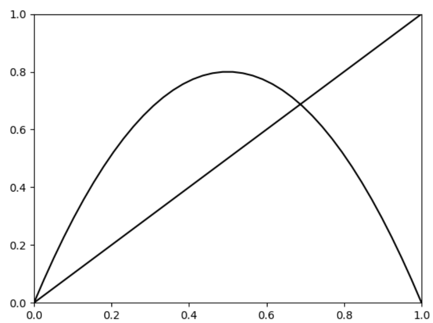
Problem 11. Draw a cobweb diagram for f_r at $r = 0.8, 1.6, 3.2, 3.5$, with one or more different starting values of $0 < x < 1$. In each of the diagrams, say whether or not the fixed points are stable.



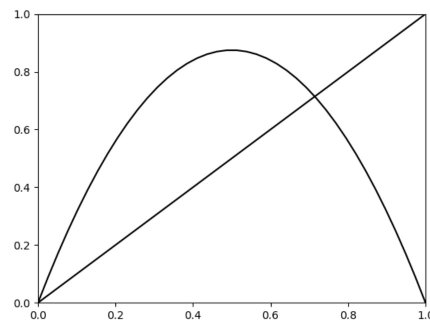
(a) $r = 0.8$



(b) $r = 1.6$



(c) $r = 3.2$



(d) $r = 3.5$

Problem 12. Make a copy of the following Google Sheet: `bit.ly/ormc-logistic`. (If you're in-person, feel free to use your phone or ask an instructor to pull it up for you.) The Google Sheet calculates the same sequence $x_{t+1} = f_r(x_t)$ that you found in the cobweb diagram. By adjusting the value of r within the range you found in Problem 9.4, for what values of r does the population converge to a fixed point? For each such r, which fixed point does it converge to? Make sure your answer agrees with the previous examples in your cobweb plots.

The proof of this takes some work, so we put it in the bonus section.

For $r = 3.2$ and $r = 3.5$ as in Problem 9, the population does not converge to a fixed point. Instead, it undergoes periodic behavior: a cycle between two values of $r = 3.2$ and four values for $r = 3.5$. We want to formally define this notion.

Definition 2.2. Let $n > 0$ be a natural number and $f : \mathbb{R} \to \mathbb{R}$ be a function. Then we say that $x \in \mathbb{R}$ is a *periodic point* of f when $f^n(x) = \underbrace{f(f(\ldots(f}_{n \text{ times}}(x))\ldots) = x$. We say that x *has period* n if n is the least positive number such that this is true.

For instance, under the map $f(x) = -x$, every point is periodic, and every point has period 2 except for 0 which has period 1. A point of period 1 is the same as a fixed point.

Problem 13. What are some values of r that have periodic points that are not fixed points? What periods do they have? Use the Google Sheet to help you.

Problem 14. For values of r close to 4, does the population continue to exhibit periodic behavior, or does something else happen?

2.1 Optional Bonus Problems

The problems here build up the following theorem.

Theorem 2.1. For $0 \leq r \leq 1$, 0 is the unique fixed point and is stable. For $1 \leq r \leq 3$, 0 is unstable and $1 - \frac{1}{r}$ is stable. For $3 < r < 4$, no fixed point is stable.

Problem 15. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant polynomial. Recall that the finitely many zeroes of f split $\mathbb{R}$ into intervals on which f is positive and intervals on which f is negative. Show that for a fixed point x_0 to be stable, $f(x) - x$ must be positive immediately to the left of x_0 or negative immediately to the right of x_0.

Problem 16. Prove that 0 is a stable fixed point for $0 \leq r \leq 1$, and that it is unstable for $1 < r \leq 4$.

We will find the following result about limits useful:

Problem 17. Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers, let $0 \leq r < 1$, and let a be a number. Show that if for all n, $|a_{n+1} - a| < r|a_n - a|$, then $a_0, a_1, a_2, \ldots$ converges to a.

Problem 18. Let $f(x) = ax^2 + bx + c$ be a quadratic, and let x_0 be a fixed point of $f(x)$. Show that if $|2ax_0 + b| < 1$, then x_0 is stable. (Hint: use the previous problem.)

Problem 19. Let $f(x) = ax^2 + bx + c$ be a quadratic, and let x_0 be a fixed point of $f(x)$. Show that if $|2ax_0 + b| > 1$, then x_0 is unstable.

Problem 20. Finish the proof of Theorem 2.1.