Voting Theory

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1 Voting Systems

In this worksheet, we're going to take a mathematical look at elections. Elections with two candidates are pretty easy to understand. We have a set V of voters, we have our two candidates, and if every voter picks their favorite candidate, we need a system for deciding who wins the election. There's only one system that seems sensible - find out which candidate got the most votes, and they win. If there's an even number of voters, there could be a tie, but other than that, this is the best way to do it. But with more candidates, it gets more complicated (see any US Presidential Election with more than two major candidates). As several US jurisdictions are currently experimenting with different voting systems (Maine, New York City), let's take a look at the options. We'll think about what makes a voting system good or fair, and see if there's a best way to do many-candidate elections.

Definition 1. Let V be a set of voters, and C a set of candidates.

A *preference profile* is an ordering of the candidates.

A *voting system* for V and C is a function that takes as input a preference profile for each voter, and outputs a single preference profile, called the *outcome* or *result*.

Problem 1. 100 voters participated in an election with five candidates, A, B, C, D, E. They each ranked the five candidates, and the ballots came in as follows:

Number of voters	Preference Profile
31	$A > D > C > E > B$
20	$B > D > C > E > A$
19	$D > C > E > B > A$
16	$E > C > B > A > D$
14	$C > E > D > B > A$

Discuss possible voting systems that could be used for this election, and in each case, find out who would win. Can you find a fair voting system that makes each candidate win?

Problem 2. One criterion for a voting system is called *Pareto Efficiency* (PE): If there are two candidates, A and B, such that every voter thinks $A > B$, then the result has $A > B$.

Which of your voting systems satisfy this criterion? Explain why this might be a good criterion to have.

Problem 3. Another criterion for a voting system is called the *Independence of Irrelevant Alternatives* (IIA): Say there are two candidates, A and B, and in a particular election, $A > B$. If the voters change some of their preferences, but not the relative position of A and B, then the result still has $A > B$.

Which of your voting systems satisfy this criterion? Explain why this might be a good criterion to have.

Problem 4. What other criteria do you think a voting system needs to be fair? Which of our voting systems satisfy these criteria?

One more criterion: A system where the result only depends on one particular voter's preference profile is called *a dictatorship*. For very obvious reasons, we'd very much like to avoid those.

2 Ultrafilters

In the standard majority-rule voting system with two candidates, our voting system works in the following simple way: We split the set of voters into two disjoint sets, based on who they rank first. Then we compare these sets, see which one is larger, and go with their opinion. In this section, we'll try to see if we can generalize this description to work with more candidates.

In order to extend this to three candidates, we will similarly start by dividing the voter set into three disjoint subsets based on their favorite candidates - then we just need to decide which set will win. This leads us to a definition:

Definition 2. An *ultrafilter* on a set V is a set U of subsets of V such that any time V is partitioned into three disjoint parts, V_1, V_2, V_3, exactly one of V_1, V_2, and V_3 is in U.

Each ultrafilter U gives rise to a voting system on three candidates, as we can just divide the voters into three sets based on their favorite candidate, and exactly one set will be in U. We say that the candidate preferred by that set is the winner. (Technically we also need to decide the rest of the ranking, but we'll get to that later.) If we understand ultrafilters, that will help us understand the limitations of voting theory with three (or more) candidates. To get a bit of practice with this definition, let's look at the simplest kind of ultrafilter as an example:

Problem 5. Let V be a set, and $v \in V$ an element. Define U_v to be the set of all subsets of V that contain v. Check that U_v is an ultrafilter - we call any ultrafilter of this form *principal*.

Problem 6. Let U be an ultrafilter on V. Show that:

- $V \in U$
- $\emptyset \notin U$
- If V_1, V_2 are disjoint sets such that $V_1 \cup V_2 = V$, then exactly one of V_1, V_2 is in U.

This shows that we can also use U to choose a winner out of two candidates.

Problem 7. Let U be an ultrafilter on V.

- Let V_1, V_2 be disjoint subsets of V such that $V_1 \cup V_2 \in U$. Show that exactly one of V_1 and V_2 is in U.
- Show by induction on n that if $V_1, \ldots, V_n$ are disjoint subsets of V such that $V_1 \cup V_2 \cup \cdots \cup V_n = V$, then exactly one V_i is in U.
- Describe how you can use U to decide not only a winner, but a ranking of any (finite) number of candidates.

Problem 8. Let U be an ultrafilter on V, and let C be a finite set of candidates. Show that the voting system based on U satisfies PE and IIA.

Problem 9. Show that every ultrafilter on a finite set V is principal, and that the voting system based on that ultrafilter is a dictatorship.

2.1 Decisive Sets

Fix a voting system on V, with candidates C. A subset $X \subseteq V$ is called *decisive* if whenever every voter in X has the same preference profile, that preference profile is the outcome of the election.

Problem 10. Let C be a set of only two candidates, and let V be a set of an odd number of voters. If the voting system is standard majority rule, which subsets of V are decisive?

Problem 11. Let U be an ultrafilter on V, let C be a set of candidates, and use the voting system based on U. What are the decisive sets?

Problem 12. Assume that a voting system's decisive sets form an ultrafilter. Show that the voting system based on that ultrafilter is the original voting system - that is, there is only one voting system with that particular ultrafilter as its decisive sets.

Problem 13. Later on, we will show that the decisive sets in any voting system with at least three candidates that satisfies PE and IIA form an ultrafilter. Taking this as an assumption for now, prove *Arrow's Impossibility Theorem*: Any voting system with at least three candidates and finitely many voters satisfying PE and IIA is a dictatorship.

The rest of the worksheet is dedicated to showing that the decisive sets in any voting system with at least 3 candidates that satisfies PE and IIA form an ultrafilter. To do this, we'll first show that with only 2 or 3 voters, there is a dictator. Then we'll consider a larger voter base, and consider a partition of the voters into three pieces. We'll show that exactly one of them is decisive, by using our special case with only three voters.

To do that, we need a more specific definition. Let a, b be candidates. We say that a set X is *decisive for* (a, b) when any time every voter in X agrees that $a < b$, the outcome of the election also has $a < b$. Note that order matters here - it is not the same for X to be decisive for (b, a).

Problem 14. Show that with two voters in a system satisfying PE and IIA, if a, b are distinct candidates, there is a voter v such that $\{v\}$ is decisive for (a, b) or $\{v\}$ is decisive for (b, a).

Problem 15. Show that with two voters in a system satisfying PE and IIA, if a, b, c are distinct candidates, then if $\{v\}$ is decisive for (a, b), then it is also decisive for (a, c) and for (b, c). Use this to show that $\{v\}$ is decisive in general, and thus that this system is a dictatorship.

Problem 16. Consider a system with three voters $\{v_1, v_2, v_3\}$, satisfying PE and IIA, with at least three candidates. Define a new system on $\{v_1, v_2\}$ by assuming v_3 copies v_2's ballot, and finding the outcome of the three-voter system. Show that if $\{v_1\}$ is decisive in that two-voter system, then $\{v_1\}$ is decisive in the original three-voter system.

Problem 17. Keep considering a system with three voters satisfying PE and IIA, with at least three candidates. Show that there is a dictator.

Hint: Show by contradiction that there is some voter v such that if you group the other two voters together as in the last problem, $\{v\}$ is decisive.

Problem 18. Consider a voting system with at least three candidates satisfying PE and IIA. Show that the set of decisive candidates is an ultrafilter, finishing the proof of Arrow's Theorem.

Hint: You have to show that for every partition of the set of voters V into three disjoint pieces $V_1 \cup V_2 \cup V_3 = V$, that exactly one of V_1, V_2, and V_3 is decisive. To do this, first construct a voting system on three voters $\{v_1, v_2, v_3\}$, each of which represents one of our three sets, with everyone voting the same way. One of these voters must be a dictator. Show that the corresponding set in the partition is decisive in the original voting system.

3 References

Problem 1 is due to Alfonso Gracia-Saz.

For this approach to Arrow's Impossibility Theorem, my main source was this post. (Although be careful reading it, I think there may be an error of their proof of Lemma 16 - the voting system $S_{v,ab}$ doesn't satisfy unanimity (what we called PE), but they assume it does.)