Set Topology

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In our physical world, we generally understand the notion of distance. In particular, let's focus on what is moment when someone says something is *close*. Being *close* however, doesn't mean anything if you don't know what you're *close to*.

Our goal will be to generalize the notion of *closeness*. We will first get some preliminary definition, then analyze the physical distance, then try to generalize on a grand scale.(i.e. in a way which can be extended to general mathematical objects)

1 Sets

Definition.
Let A and B be sets. The *union* $A \cup B$ of two sets A and B is the set of elements which are in A, in B, or in both A and B.

Definition.
Let A and B be sets. The *intersection* $A \cap B$ is the set of elements which are in both A and B.

Definition.
Suppose that we speak of some kind of mathematical space denoted by some tuple $(X, \ldots)$. Here X is a specific set where the objects we consider all live in. In practice, what comes after the X can be any number of qualities or defining functions that are associated with the kind of space the tuple is. This X is sometimes called *the universe*.

Definition.
Let $A \subset X$ be the subset of some universe X, then the *complement* of A is the set containing all the elements in X which are not in A. This set is denoted by $X \setminus A$. When the containing set is clear, we can write A^c as a simplified notation for $X \setminus A$.

Note: The complement of a set does not have to be used in context of a universe if the set containing A is specified. For example, if $A \subset B \subset X$, then A^c would generally be interpreted as the set $X \setminus A$. However, if we say "A^c with respect to the set B", then this would be set $B \setminus A$ instead.

Problem 1. (a) Show that $(A \cup B)^c = A^c \cap B^c$.

(b) Show that $(A \cap B)^c = A^c \cup B^c$.

Problem 2. (a) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 3 (*)**.** (a) Generalise the notions of union and intersection to arbitrary (possibly infinite) collection of sets Γ.

(b) Show that

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} A \cap B$$

.

(c) Show that

$$A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} A \cup B$$

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2 Topological Space

We can now use the properties of the physical distance in $\mathbb{R}$ (also called the standard metric in $\mathbb{R}$) as an inspiration for the general notion of closeness in an abstract space.

Let 2^X be the set of **all** subsets of X, also known as the *power set* of X.

Definition.
Let $\mathcal{T} \subset 2^X$ be a subset of the power set of X, so $\mathcal{T}$ has elements which are subsets of X. We call $\mathcal{T}$ a *topology* on X if the following is true:

- $\emptyset \in \mathcal{T}$,
- $X \in \mathcal{T}$,
- If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$,
- If $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of sets in $\mathcal{T}$, where α is taken from possibly infinite index set $\mathcal{I}$ (like $\mathbb{N}$ or $\mathbb{R}$), then $\bigcup_{\alpha \in \mathcal{I}} U_\alpha \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called the *open sets* of X.

Problem 4.
Check that the following are topologies of X.

(a) The discrete topology $\mathcal{T} = 2^X$.

(b) The indiscrete topology $\mathcal{T} = \{\emptyset, X\}$.

Problem 5 (*)**.**
Prove that the following are topologies:

(a) $X = \{0, 1\}$, $\mathcal{T} = \{\emptyset, \{0\}, \{0, 1\}\}$;

(b) $X = \mathbb{R}$, $\mathcal{T}$ consists of the empty set, $\mathbb{R}$ and all rays of the form $(a, +\infty)$.

Problem 6 (*).
Prove that the following are NOT topologies:

(a) $X = \mathbb{R}$, $\mathcal{T}$ consists of the empty set and all infinite subsets of $\mathbb{R}$;

(b) $X = \mathbb{R}$, $\mathcal{T}$ consists of the empty set, $\mathbb{R}$ and all rays of the form $[a, +\infty)$.

Definition.
An open ball $B_r(a)$ on a line is a set of points lying within distance less than r from the point a.

On a real line this notation is equivalent to $(a-r, a+r)$, but it is different in different metrics, which you can know from the next section.

Problem 7.
Let $\mathcal{T}_{std} \subset 2^X$ be given by

$$\mathcal{T}_{std} \;=\; \{\emptyset\} \,\cup \{B_r(a)\} \;\cup\; \{B \mid B = \bigcup_{\alpha \in \mathcal{I}} B_{\epsilon_\alpha}(y_\alpha)\}.$$

In other words, the open sets are the empty set, open balls in $\mathbb{R}$ along with sets which are unions of open balls. Prove that $\mathcal{T}_{std}$ is a topology on $\mathbb{R}$. This topology is called the *standard topology on* $\mathbb{R}$.

Definition.
If V is an open set of X and $x \in V$, then we call V a *neighborhood* of x.

Problem 8.
Let $X = \mathbb{R}$, find two neighborhoods of the point $0 \in \mathbb{R}$ in the following topologies (if there aren't two, find one).

(a) In the standard topology $\mathcal{T}_{std}$,

(b) In the discrete topology $\mathcal{T}_{disc}$,

(c) In the indiscrete topology $\mathcal{T}_{ind}$.

The concept of open sets and neighborhoods does in fact generalize closeness. We can say that points $x, y \in X$ are V*–close* if V is a neighborhood of both x and y, and this relates back to our physical distance. Notice that, if x, y, z are real numbers such that x and y are contained in $V = B_{\epsilon/2}(z)$, then x and y are V-close, or in this case ϵ-close in the standard topology over the reals. Notice this is equivalent to saying $|x - y| < \epsilon$.

3 Distance

X can then be associated with a line, a plane or a space. We want to find the subsets of X which we can use to define being *close* to any individual position. More explicitly, we want to define a subset $B_\epsilon(y) \subset X$ such that all the points in $B_\epsilon(y)$ are within a distance ϵ away from a point y.

Let X be any set. It is sometimes convenient to think of X as a real line ($\mathbb{R}$) or a plane ($\mathbb{R}^2$) and call its elements "points".

Definition.
A function $d : X \times X \to \mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ (reads as function d from pairs of points to nonnegative numbers) is called a metric (or distance function) in X if

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ for any $x, y \in X$;
3. $d(x, y) \geq d(x, z) + d(z, y)$ for any $x, y, z \in X$.

Problem 9.
Check that the function

$$d(x, y) : X \times X \to \mathbb{R}_+ : (x, y) \mapsto \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y, \end{cases}$$

is a metric for any set X

Problem 10.
Check that the function

$$d(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$$

is a metric on $\mathbb{R}$.

Problem 11.
Check that the function

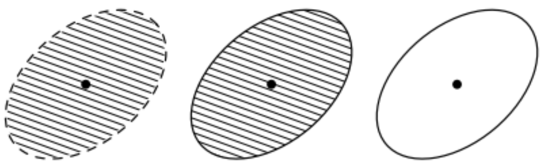
$$d(x, y) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+ : (x, y) \mapsto \text{distance from } x \text{ to } y$$

is a metric on $\mathbb{R}^2$.

Definition.
Let X be a set, d a metric on this set, $a \in X$ a point, r a positive real number. Then the sets

$$B_r(a) = \{x \in X | d(a, x) < r\},$$
$$D_r(a) = \{x \in X | d(a, x) \leq r\},$$
$$S_r(a) = \{x \in X | d(a, x) = r\}$$

are, respectively, called the open ball, closed ball, and sphere with center a and radius r.



Problem 12.
Check that in Euclidean metric on $\mathbb{R}$ any closed ball is a segment and any sphere is a pair of points, on $\mathbb{R}^2$ any sphere is a circle and on $\mathbb{R}^3$ any ball is a ball and any sphere is a sphere.

Problem 13.
Prove that for any points x and a of any metric space and any $r > d(a, x)$ we have $B_{r-d(a,x)}(x) \subset B_r(a)$ and $D_{r-d(a,x)}(x) \subset D_r(a)$.

Problem 14. (a) Check that the function

$$d(x, y) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+ : ((x_1, y_1), (x_2, y_2)) \mapsto |x_1 - x_2| + |y_1 - y_2|$$

is a metric on $\mathbb{R}^2$. It is called Manhattan distance.

(b) Draw some ball in that metric.

Problem 15 (*).
Find a finite set X, some metric on it and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.