The Complex Numbers and Geometry I

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1 The Complex Numbers

Two weeks ago, we saw that the function $f(x) = x^2$ when considered as a function into the real numbers $\mathbb{R}$ is not surjective. This is because negative numbers are not in its image - or equivalently to say, there is no square root of -1 in the real numbers $\mathbb{R}$. It is therefore natural to toss $\sqrt{-1}$ into our number system - let's see what happens.

Definition 1 *An* ***imaginary number*** *is a real multiple of $\sqrt{-1}$. A* ***complex number*** *is the sum of a real number and an imaginary number; that is to say, a complex number z has the form*

$$z = a + b\sqrt{-1} \text{ where } a, b \text{ are real numbers.}$$

The set of complex numbers is denoted $\mathbb{C}$. Complex variables are usually written with the letters z and w rather than x and y, which are used for real variables.

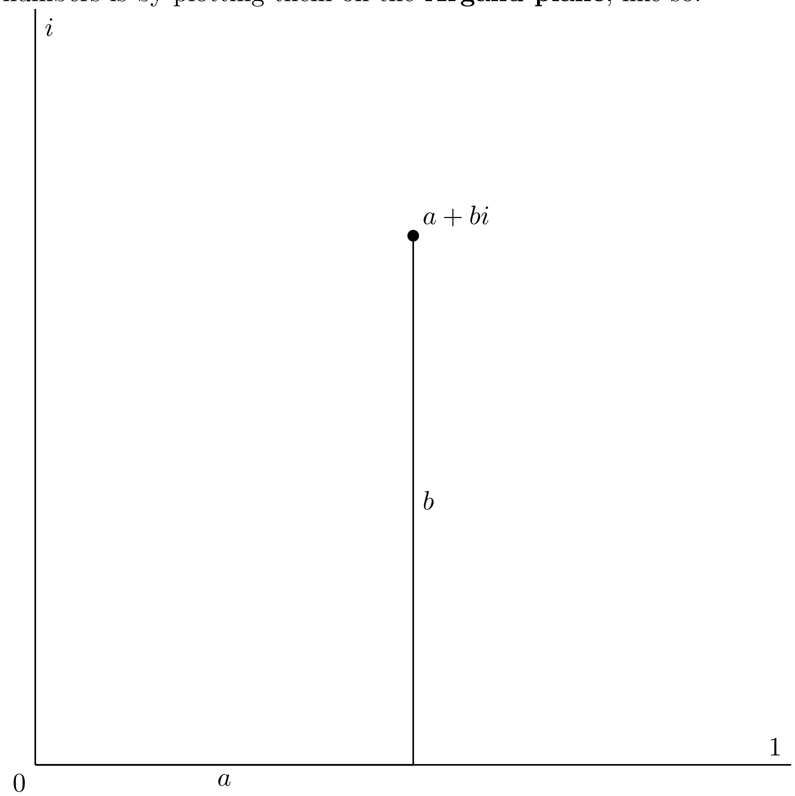
We'd like to be able to do the same things we do with real numbers - add, subtract, multiply, and divide. For now, though, let's focus on addition and multiplication - the others will come later.

Problem 1 *Compute the following (hint: you can treat the $\sqrt{-1}$ like a variable and do familiar operations from algebra, like combining like terms, FOILing, etc.)*

- $(1 + \sqrt{-1}) + (2 + \sqrt{-1})$
- $(3 + 2\sqrt{-1}) + (3 - 2\sqrt{-1})$
- $(1 + \sqrt{-1}) \times (2 + \sqrt{-1})$
- $(3 + 2\sqrt{-1}) \times (3 - 2\sqrt{-1})$

2 The Argand Plane

Since $\sqrt{-1}$ is quite tiresome to write all the time, we usually denote it with the letter i. Another way to look at complex numbers is by plotting them on the **Argand plane**, like so:



Instead of x and y axes, the Argand plane has 1 and i axes. More importantly though, it lets us do something we can't do in the xy-plane, which is to add and subtract points.

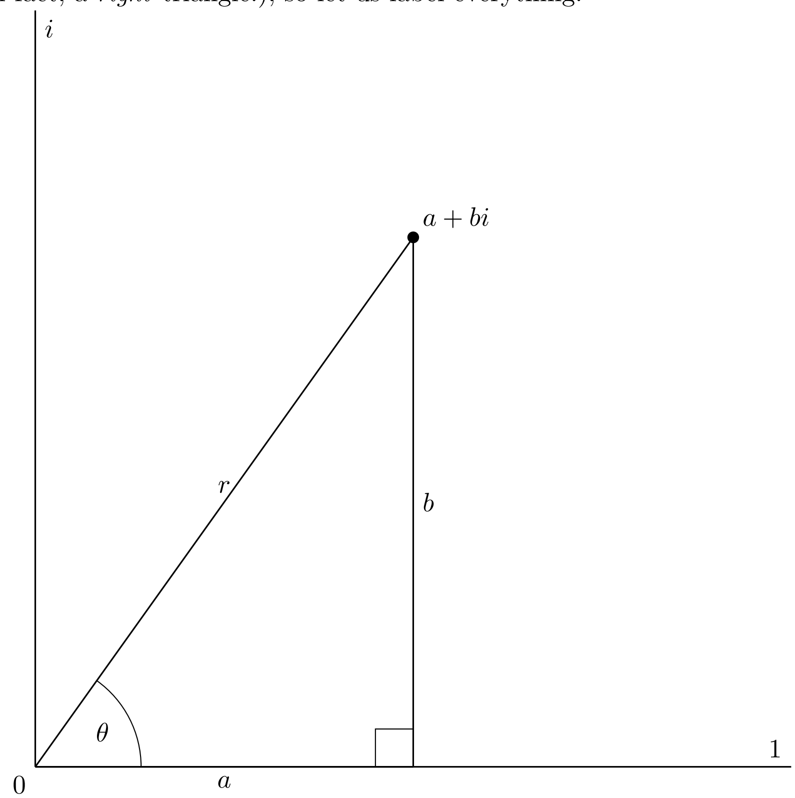
Problem 2 *Plot the points* 0, $(1+i)$, $(2+i)$, *and* $(1+i)+(2+i)$ *in the space below. Do you notice anything? (Hint: try connecting the points with line segments)*

Problem 3 *Plot the points* 0, $(1+i)$, $(2+i)$, *and* $(1+i)\times(2+i)$ *in the space below. Do you notice anything?*

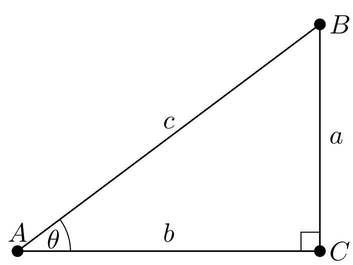
Problem 4 *(Challenge) Prove anything you noticed in the previous two problems.*

3 The Polar Form and Trigonometry

In the diagram we had earlier, we can draw a straight line from 0 to $a + bi$. This makes a triangle with the lines we drew (in fact, a *right* triangle!), so let us label everything:



In order to describe the relationship between a, b, r, θ, we will introduce trig functions, which are first defined on a right triangle:

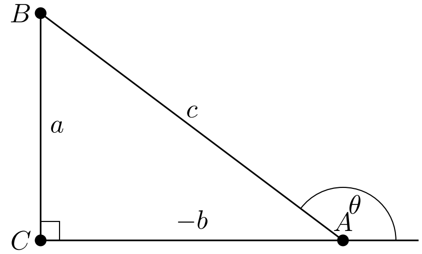


Definition 2 *The functions **sine** and **cosine** of the angle θ are defined by*

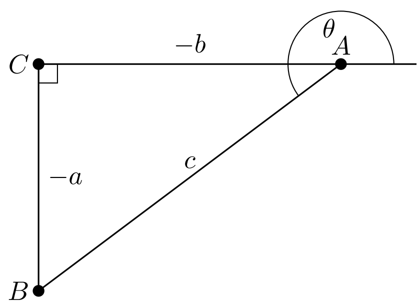
$$\sin(\theta) = \frac{opposite}{hypotenuse} = \frac{a}{c} \text{ and } \cos(\theta) = \frac{adjacent}{hypotenuse} = \frac{a}{c}$$

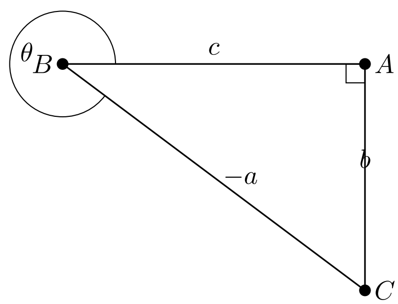
This definition appears to only be good for angles between 0 and $\frac{\pi}{2}$ radians, but if we draw the right picture, we can extend it to all angles.

If θ is between $\frac{\pi}{2}$ and π, we consider the adjacent side to be $-b$ long instead, because if this were in the xy-plane, it would be going in the *negative* x direction. We define $\sin(\theta)$ and $\cos(\theta)$ accordingly.



Finally, we see from the following pictures that sine and cosine are both negative for θ between π and $\frac{3\pi}{2}$, and that sine is negative and cosine is positive for θ between $\frac{3\pi}{2}$ and 2π.





We'll also define the other trig functions based on sine and cosine.

Definition 3 *The functions* ***tangent****,* ***secant****,* ***cosecant****, and* ***cotangent*** *of the angle θ are defined by*

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \textit{ and } \sec(\theta) = \frac{1}{\cos(\theta)} \textit{ and } \csc(\theta) = \frac{1}{\sin(\theta)} \textit{ and } \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

Also, the function $(\cos(\theta))^2$ is usually written $\cos^2(\theta)$, and similarly for the other trig functions.

Problem 5 *Given a complex number in the form (r, θ), write it as $a + bi$ for some a, b in terms of r, θ, and trig functions.*

Problem 6 *Given a complex number in the form $a + bi$, write it as (r, θ) for some r, θ in terms of a, b, and trig functions.*

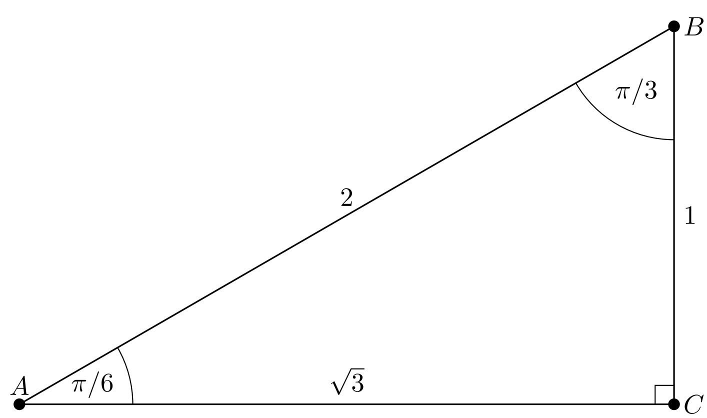
We have just seen that specifying $z = a + bi$ is the same as specifying $z = (r, \theta)$.

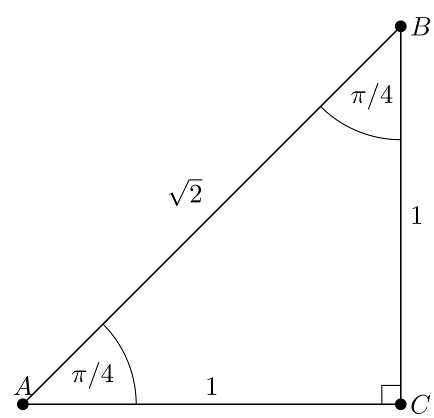
Definition 4 *$z = a + bi$ is called the* ***rectangular form*** *of z, while $z = (r, \theta)$ is called the* ***polar form*** *of z.*

For the rectangular form, a is called the ***real part*** *and b (not bi!) is called the* ***imaginary part*** *of z, denoted $Re(z)$ and $Im(z)$, respectively.*

For the polar form, r is called the ***modulus*** *(or sometimes* ***magnitude****) of z, while θ is called the* ***argument*** *of z, denoted $|z|$ and $Arg(z)$, respectively. Note that θ is only defined up to multiples of 2π.*

In order to switch between rectangular and polar forms, it helps us to remember some important right triangles to compute sines and cosines. (the following triangles are not drawn to scale)





Problem 7 *Convert the following complex numbers from rectangular form to polar form.*

- i

- $1 + i$

- (Challenge) $(\sqrt{6} + \sqrt{2}) + (\sqrt{6} - \sqrt{2})i$

Problem 8 *Convert the following complex numbers from polar form to rectangular form.*

- $(1, 3\pi/2)$

- $(2, \pi/6)$

- (Challenge) $(5, 2\tan^{-1}(1/3))$

4 Proving Trig Identities

As you may have seen, switching from rectangular to polar is a bit cumbersome. To prepare for next week, let us prove some trig identities to make our lives much easier.

Problem 9 *Prove the two identities*

$$\cos(\frac{\pi}{2} - \theta) = \sin(\theta) \textit{ and } \sin(\frac{\pi}{2} - \theta) = \cos(\theta)$$

(Hint: It comes down to drawing the right picture. Remember that the angles of every triangle in the plane add up to π!)

Problem 10 *Prove that*

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

(Hint: Once again, it comes down to drawing the right picture. This one is very hard though, so don't be afraid to ask your instructors/parents for help!)

Problem 11 *Prove that*

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

(Hint: Once again, you could draw a picture. But might there be an easier way, using another identity we just showed?)

5 Bonus Section: More Trig identities

Problem 12 *If you're done, go back and try some of the challenge problems. Alternatively, here's some more trig identities you can try to prove (you'll have to draw these out on your own piece of paper).*

- $$\cos^2(\theta) + \sin^2(\theta) = 1$$
- $$\sec^2(\theta) = 1 + \tan^2(\theta)$$
- $$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
- $$\cos(2\theta) = 1 + 2\cos^2(\theta)$$
- $$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$
- $$\sin(\frac{1}{2}\theta) = \sqrt{\frac{1 - \cos^2(\theta)}{2}}$$
- $$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$