ORMC Olympiad Group
Week 6
Inequalities III: Cauchy-Schwartz

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November 7, 2021

Theorem 1 (Cauchy-Schwartz Inequality). *Let* $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n \in \mathbb{R}$*. Then the following inequality holds*

$$(x_1^2 + x_2^2 + \ldots x_n^2)(y_1^2 + y_2^2 + \ldots y_n^2) \geq (x_1y_1 + x_2y_2 + \ldots x_ny_n)^2$$

Equality holds when $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$.

Problems

1. x, y are real values so that $x^2 + y^2 = 4$. Find the maximum value that $x + 2y$ can take.
   (a) Solve using quadratics.
   (b) Solve using Cauchy-Schwartz Inequality.
   (c) Solve using geometry.
2. (a) Let a, b, c be positive reals. Prove

$$(a + b)(a + c)(b + c) \geq 8abc$$

(b) **(PSS)** $n \geq 3$ is a positive integer. Let $x_1, x_2, \ldots, x_n$ be reals with $x_1 \cdot x_2 \cdots x_n = 1$. Prove that $(x_1+1)(x_2+1)\ldots(x_n+1) \geq 2^n$

3. x, y are two real numbers whose sum of squares is 1. Find the maximum and minimum possible value of $3x + 4y$.

4. $x^2 + y^2 + z^2 = 1$. Find the maximum possible value of $3x + 4y + 12z$. Give the values when max occurs.

5. **(TJNMO-FR 2011)** For which of the (A, B) pairs, there is no real solution to the equations

$$2x + y = A, \qquad x^2 + y^2 = B?$$

(A) $(\frac{5}{9}, \frac{9}{7})$ (B) $(1, \frac{2}{9})$ (C) $(\frac{4}{3}, \frac{1}{3})$ (D) $(\frac{9}{5}, \frac{2}{3})$ (E) $(2, \frac{6}{7})$

6. **(AHSME 1993)** Which of the following could NOT be the lengths of the external diagonals of a right regular prism [a "box"]? (An *external diagonal* is a diagonal of one of the rectangular faces of the box.)

(A) $\{4, 5, 6\}$ (B) $\{4, 5, 7\}$ (C) $\{4, 6, 7\}$ (D) $\{5, 6, 7\}$ (E) $\{5, 7, 8\}$

7. Let a, b, c be positive reals. Prove $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$

8. Positive real values x, y satisfy the equality $x^3 + y^4 = x^2 y$. Let A be the maximum value that x can take, and B be the maximum value that y can take. $\frac{A}{B}$ can be represented as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $\frac{A}{B}$

9. Let a, b, c be positive reals. Prove the following:

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

Give an example when the equality occurs.

10. **(AIME 2009II)** Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = \frac{3}{5}$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in

the form $r - s\sqrt{t}$, where r, s and t are positive integers and t is not divisible by the square of any prime. Find $r + s + t$.

11. **(LAMC 2008)** Is it true that if all the sides of a triangle are less than 1 then the area is less than $\sqrt{3}/4$?