ORMC Olympiad Group
Week 5 Solutions

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November 5, 2021

Solutions

Solution 1

We have $x = y + 2$. So, the expression is $(y+2)^2 - 2y^2 = -y^2 + 4y + 4$. Then we can rewrite this as

$$-y^2 + 4y - 4 + 8 = -(y-2)^2 + 8,$$

so the maximum value is $\boxed{8}$.

Solution 2

Let $s = x + y$ and $t = xy$. Then, expanding the given equation gives

$$x^2y^2 + x^2 + y^2 + 1 + 9 = 6(x+y) \implies t^2 + s^2 - 2t + 1 + 9 = 6s.$$

(Note that $x^2 + y^2 = (x+y)^2 - 2xy$). This equation rearranges to

$$s^2 - 6s + 9 + t^2 - 2t + 1 = 0 \implies (s-3)^2 + (t-1)^2 = 0.$$

Thus $s = 3$ and $t = 1$, so $x^2 + y^2 = s^2 - 2t = \boxed{7}$.

Solution 3

Substitute $z = 12 - 3x - 2y$, so the expression becomes

$$x^3 + y^2 + 12 - 3x - 2y = (x^3 - 3x) + (y^2 - 2y) + 12.$$

We know $x^3 - 3x \geq -2$ (problem 6a from last week) and $y^2 - 2y \geq -1$ (expand $(y-1)^2 \geq 0$), so

$$(x^3 - 3x) + (y^2 - 2y) + 12 \geq -2 - 1 + 12 = \boxed{9}.$$

Solution 4

(a) The fraction can be factored as $\dfrac{(x-1)(x+1)}{(x-2)(x+2)}$, so we divide the number line into intervals separated by the numbers $-2, -1, 1, 2$ (i.e. roots of the numerator and denominator). Upon testing the sign in each of those intervals, we can see that it is negative for $x \in \boxed{(-2,-1) \cup (1,2)}$.

(b) Note that $x^2 \geq 0$ is always satisfied, so this term will not change the sign of the expression. Then, note that this is just the product of $(x^2 - 1)$ and $(x^2 - 4)$ and the expression in part (a) was the quotient of these; they should have the same sign everywhere though, so the answer must still be $\boxed{(-2,-1) \cup (1,2)}$.

(c) The (real) roots of the expression are clearly just $\pm 1, \pm 2$. Thus we split up the number line into intervals with these numbers, and testing the sign in each interval gives that the expression is nonpositive for $x \in \boxed{(-\infty,-2] \cup [-1,2]}$.

(d) The only (real) roots of this expression are ± 1; so we split the number line into three intervals divided by ± 1, and test signs; the expression is negative in $x \in \boxed{(-\infty,-1) \cup (-1,1)}$.

(e) First note that we can ignore the $x^2 + 4$ and x^2 terms as these are always nonnegative. Then we divide the number line into intervals split by $\pm 1, 2, 3, 5$, the roots of the numerator and denominator. We can see that the expression starts positive in the most negative interval, and then alternates between negative and positive. Thus the sum of the interval lengths is $2 + 1 + 1 + 2$ (i.e. $(-5,-3) \cup (-2,-1) \cup (1,2) \cup (3,5)$) which is $\boxed{6}$.

Solution 5

The inequality rearranges to $2x^2 + ax - b < 0$. We know this must also be $2(x+1)(x-2018) < 0$ based on the solution set, so $a = 2(1-2018)$ and $b = 2(-(1)(-2018))$. Thus $a + b = \boxed{2}$.

Solution 6

The function factors as $f(x) = \dfrac{(x-1)(x-m+1)}{(x-m)(x+m)}$. So, we draw our inequality line with ranges split by $-m, 1, m-1, m$ and check the signs. we see that $S = (-m, 1) \cup (m-1, m)$. Thus the sum of the lengths is $m + 1 + 1 = 15 \implies m = \boxed{13}$.

Solution 7

Note that

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 &= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1) \\ &= (x_1 + x_2 + x_3)((x_1 + x_2 + x_3)^2 - 3(x_1x_2 - x_2x_3 - x_3x_1)) \\ &= a(a^2 - 3a) \\ &= a^2(a-3). \end{aligned}$$

So, equivalently, we will maximize $a^2(3-a)$. Note that $a^2(3-a) = 4\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)(3-a)$, and then by AM-GM,

$$\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)(3-a) \le \left(\frac{\frac{a}{2} + \frac{a}{2} + (3-a)}{3}\right)^3 = 1.$$

Thus the answer is $4(1) = \boxed{4}$.

Solution 8

We repeatedly complete the square in this expression. For example, consider the terms $x^2 - xy$. To complete the square for this, we would need a $+\dfrac{y^2}{4}$; so we split up the y^2 term into $\dfrac{y^2}{4}$ and $\dfrac{3y^2}{4}$. Now we must complete the square on $\dfrac{3y^2}{4} - yz$, and so on. The expression becomes

$$\left(x^2 - xy + \frac{y^2}{4}\right) + \left(\frac{3y^2}{4} - yz + \frac{z^2}{3}\right) + \left(\frac{2z^2}{3} - zt + \frac{3t^2}{8}\right) + \left(\frac{5t^2}{8} - 10t + 40\right) - 40.$$

Thus the answer is $\boxed{-40}$.

Solution 9

(a) Note that $p(-4) = -9 < 0$ and $p(-2) = 1 > 0$. Then since p is clearly continuous, there must be a root between -4 and -2 (intermediate value theorem). Similarly, since $p(0) = -5 < 0$, there must be another root between -2 and 0. Then since $p(x)$ grows arbitrarily large (positive) for larger x, there should be another root, which is some positive number.

(b) We know the roots are r_1, r_2, r_3 for some $r_1 \in (-4, -2)$, $r_2 \in (-2, 0)$, and $r_3 > 0$. Thus we can split the number line by the r_i and $-4, -2, 0$. Then the intervals where the expression is negative is

$$(-4, r_1) \cup (-2, r_2) \cup (0, r_3).$$

Thus the sum of the lengths is

$$(r_1 - (-4)) + (r_2 - (-2)) + (r_3 - 0) = r_1 + r_2 + r_3 + 6.$$

But $r_1 + r_2 + r_3 = -4$ from the coefficients of the polynomial, so the answer is $-4 + 6 = \boxed{2}$.

Solution 10

WLOG let $a_1 < a_2 < \cdots < a_5$. We will examine the sum $\sum_{1 \le i < j \le 5} |a_i - a_j|^2$. Note that by the given condition that each pair of a_i differs by at least 1,

$$\sum_{1 \le i < j \le 5} |a_i - a_j|^2 \ge 4 \cdot 1^2 + 3 \cdot 2^2 + 2 \cdot 3^2 + 1 \cdot 4^2 = 50.$$

But also note that

$$\sum_{1 \le i < j \le 5} |a_i - a_j|^2 = 4 \sum a_i^2 - 2 \sum a_i a_j,$$

and

$$\left(\sum a_i \right)^2 = \sum a_i^2 + 2 \sum a_i a_j \implies (2k)^2 = 2k^2 + 2 \sum a_i a_j,$$

so $2 \sum a_i a_j = 2k^2$. Thus

$$\sum_{1 \le i < j \le 5} |a_i - a_j|^2 = 4 \sum a_i^2 - 2 \sum a_i a_j = 4(2k^2) - 2k^2 = 6k^2.$$

Thus, $6k^2 \ge 50 \implies 30k^2 \ge \boxed{250}$.