

ORMC Olympiad Group  
Week 5  
Inequalities II: AM-GM Continued &  
Inequality Intervals

Osman Akar

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## Problems

1. Let  $x > y$  are two real numbers whose difference is 2. Find the maximum possible value of  $x^2 - 2y^2$ .
2. **(Turkey 2014)** If  $(x^2 + 1)(y^2 + 1) + 9 = 6(x + y)$  where  $x, y$  are real numbers, what is  $x^2 + y^2$ ?
3. **(TJNMO-FR 2012)**  $x, y, z$  are nonnegative real numbers with equality  $3x + 2y + z = 12$ . Find the least value that  $x^3 + y^2 + z$  can take.
4. (a) Find the set of real values of  $x$ , as a union of open intervals where  $\frac{x^2-1}{x^2-4} < 0$   
(b) Find the set of real values of  $x$ , as a union of open intervals where  $x^2(x^2 - 1)(x^2 - 4) < 0$   
(c) Find the set of real values of  $x$ , as a union of open intervals where  $(x^2 - 1)(x^3 - 1)(x^2 - 4) \leq 0$   
(d) Find the set of real values of  $x$ , as a union of open intervals where  $(x^2 - 1)(x^3 - 1)(x^4 - 1) < 0$

(e) Let  $f(x) = \frac{(x^2-1)(x^2-9)(x^2-25)}{(x^2+4)x^2(x^2-4)}$ . Find the sum of lengths of intervals where  $f(x) < 0$ .

5. **(TJNMO-FR 2018)** Find  $a + b$ , if the solution set of the inequality for  $x(2x + a) < b$  is  $(-1, 2018)$

6. For real number  $m > 2$ , the function  $f$  defined as

$$f(x) = \frac{x^2 - mx + m - 1}{x^2 - m^2}$$

for all reals  $x \neq \pm m$ . The solution set of the inequality  $f(x) < 0$  can be represented as a union of intervals  $S = (a, b) \cup (c, d)$ . If the sum of the length of the intervals is 15, what is  $m$ ?

7. **(HMMT 2007 Guts)** For  $a$  a positive real number, let  $x_1, x_2, x_3$  be the roots of the equation  $x^3 - ax^2 + ax - a = 0$ . Determine the smallest possible value of  $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$ .

8. Let  $x, y, z, t$  be real numbers. Find the minimum possible value of

$$x^2 + y^2 + z^2 + t^2 - xy - yz - zt - 10t$$

9.  $p(x) = x^3 + 4x^2 + x - 5$  is a monic cubic polynomial.

(a) Show that  $p(x)$  has 3 real roots.

**HINT: Look at values of  $P(-4), p(-2), p(0)$ .**

(b) The set of real numbers  $x$  for which

$$\frac{x^3 + 4x^2 + x - 5}{x(x+2)(x+4)} < 0$$

is the union of intervals of the form  $a < x < b$ . What is the sum of the lengths of these intervals?

10. **(2013W Junior Tubitak Camp - Refail Alizade)** Real numbers  $a_1, a_2, a_3, a_4, a_5$  satisfy  $|a_i - a_j| \geq 1$  for all  $i \neq j$ . It is given that

$$a_1 + a_2 + a_3 + a_4 + a_5 = 2k$$

and

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2$$

Find the minimal value of  $30k^2$