ORMC Olympiad Group
Week 5
Inequalities II: AM-GM Continued & Inequality Intervals

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Problems

1. Let $x > y$ are two real numbers whose difference is 2. Find the maximum possible value of $x^2 - 2y^2$.

2. (Turkey 2014) If $(x^2 + 1)(y^2 + 1) + 9 = 6(x + y)$ where $x, y$ are real numbers, what is $x^2 + y^2$?

3. (TJNMO-FR 2012) $x, y, z$ are nonnegative real numbers with equality $3x + 2y + z = 12$. Find the least value that $x^3 + y^2 + z$ can take.

4. (a) Find the set of real values of $x$, as a union of open intervals where $\frac{x^2 - 1}{x^2 - 4} < 0$

(b) Find the set of real values of $x$, as a union of open intervals where $x^2(x^2 - 1)(x^2 - 4) < 0$

(c) Find the set of real values of $x$, as a union of open intervals where $(x^2 - 1)(x^3 - 1)(x^2 - 4) \leq 0$

(d) Find the set of real values of $x$, as a union of open intervals where $(x^2 - 1)(x^3 - 1)(x^4 - 1) < 0$
(e) Let \( f(x) = \frac{(x^2-1)(x^2-9)(x^2-25)}{(x^2+4)x^2(x^2-4)} \). Find the sum of lengths of intervals where \( f(x) < 0 \).

5. **(TJNMO-FR 2018)** Find \( a + b \), if the solution set of the inequality for \( x(2x + a) < b \) is \((-1, 2018)\)

6. For real number \( m > 2 \), the function \( f \) defined as
\[
 f(x) = \frac{x^2 - mx + m - 1}{x^2 - m}
\]
for all reals \( x \neq \pm m \). The solution set of the inequality \( f(x) < 0 \) can be represented as a union of intervals \( S = (a, b) \cup (c, d) \). If the sum of the length of the intervals is 15, what is \( m \)?

7. **(HMMT 2007 Guts)** For a a positive real number, let \( x_1, x_2, x_3 \) be the roots of the equation \( x^3 - ax^2 + ax - a = 0 \). Determine the smallest possible value of \( x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \).

8. Let \( x, y, z, t \) be real numbers. Find the minimum possible value of
\[
x^2 + y^2 + z^2 + t^2 - xy - yz - zt - 10t
\]

9. \( p(x) = x^3 + 4x^2 + x - 5 \) is a monic cubic polynomial.
   (a) Show that \( p(x) \) has 3 real roots.

   **HINT: Look at values of \( P(-4), P(-2), P(0) \).**

   (b) The set of real numbers \( x \) for which
\[
\frac{x^3 + 4x^2 + x - 5}{x(x + 2)(x + 4)} < 0
\]
is the union of intervals of the form \( a < x < b \). What is the sum of the lengths of these intervals?

10. **(2013W Junior Tubitak Camp - Refail Alizade)** Real numbers \( a_1, a_2, a_3, a_4, a_5 \) satisfy \( |a_i - a_j| \geq 1 \) for all \( i \neq j \). It is given that
\[
a_1 + a_2 + a_3 + a_4 + a_5 = 2k
\]
and
\[
a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 2k^2
\]
Find the minimal value of \( 30k^2 \)