ORMC Olympiad Group
Week 4 Solutions

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Solutions

Solution 1

Completing the square, we have

$$x^2 + y^2 + 4x - 6y = (x+2)^2 - 4 + (y-3)^2 - 9 \le \boxed{-13}.$$

Solution 2

(a) By AM-GM inequality,

$$\frac{x^2+1}{x} = x + \frac{1}{x} \ge 2\sqrt{x \cdot \frac{1}{x}} = \boxed{2}.$$

Equality occurs when $x = \frac{1}{x}$, so when $\boxed{x=1}$.

(b) By AM-GM,

$$\frac{12x^2+3}{x} = 12x + \frac{3}{x} \ge 2\sqrt{12x \cdot \frac{3}{x}} = \boxed{12}.$$

Equality occurs when $12x = \frac{3}{x}$, so when $\boxed{x = \frac{1}{4}}$.

(c) We rewrite 162 as $81 + 81$ and then use AM-GM with 4 terms:

$$x^4 + y^4 + 81 + 81 \ge 4\sqrt[4]{x^4y^4(81)(81)} = 36xy.$$

Thus, $\frac{x^4+y^4+162}{xy} \ge \boxed{36}$. Equality occurs when $x^4 = y^4 = 81$, so when $\boxed{x=y=3}$.

(d) Following a similar strategy as in part (c),

$$\frac{x^4+y^4+\frac{1}{2}+\frac{1}{2}}{xy} \ge \frac{4\sqrt[4]{x^4y^4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}}{xy} = \boxed{2\sqrt{2}}.$$

Equality occurs when $x^4 = y^4 = \frac{1}{2}$, so when $\boxed{x = y = 2^{-\frac{1}{4}}}$.

Solution 3

Note that

$$\begin{aligned} xy + z &= (x+z)(y+z) \\ \implies xy + z &= xy + z(x+y) + z^2 \\ \implies 1 &= x + y + z \end{aligned}$$

(we can divide by z in the final step since it is positive). Then, by AM-GM inequality, $3\sqrt[3]{xyz} \le x + y + z$, so

$$xyz \le \left(\frac{x+y+z}{3}\right)^3 = \boxed{\frac{1}{27}}.$$

Equality holds when $x = y = z = \frac{1}{3}$.

Solution 4

By AM-GM,

$$\frac{9x^2\sin^2 x+4}{x\sin x}=9x\sin x+\frac{4}{x\sin x}\geq 2\sqrt{9x\sin x\cdot\frac{4}{x\sin x}}=\boxed{12}.$$

We should also check that the equality case is achievable for $x\in(0,\pi)$. Equality occurs when $9x\sin x=\frac{4}{x\sin x}\implies x\sin x=\frac{2}{3}$. Then, note that $x\sin x$ is continuous and evaluates to 0 at $x=0$ and $\frac{\pi}{2}$ at $x=\frac{\pi}{2}$. Thus by intermediate value theorem, the equality case is achieved for some $x\in\left(0,\frac{\pi}{2}\right)$.

Solution 5

We rewrite the expression as $(a)(a)(12-2a)$ and then apply AM-GM (which we can do since all terms are nonnegative for $0\leq a\leq 6$):

$$(a)(a)(12-2a)\leq\left(\frac{a+a+(12-2a)}{3}\right)^3=\boxed{64}.$$

Equality occurs when $a=12-2a$, so when $a=4$.

Solution 6

(a) We want to find a such that $a\geq 3x-x^3$ holds with an achievable equality case. Equivalently, we want the smallest a such that $a\geq 3x-x^3$ holds. Rewrite this as $x^3+a\geq 3x$, and then note that by AM-GM,

$$x^3+\frac{a}{2}+\frac{a}{2}\geq 3x\sqrt[3]{\frac{a^2}{4}},$$

with achievable equality at $x^3=\frac{a}{2}$. Then, we can see that $x^3+a\geq 3x$ would hold for all $a\geq 2$, and with equality holding for $a=2$ (and that the inequality does not hold for $a<2$). Thus the answer is $\boxed{2}$.

(b) Similarly to above, we have

$$x^3+\frac{a}{2}+\frac{a}{2}\geq 3x\sqrt[3]{\frac{a^2}{4}}.$$

We want to find the smallest a such that $x^3+a\geq 2x$; clearly this would then be the a satisfying

$$3\sqrt[3]{\frac{a^2}{4}}=2\implies a=\boxed{\frac{4\sqrt{6}}{9}}.$$

Solution 7

(a) We apply AM-GM to every pair of variables:

$$a^2+b^2\geq 2ab,\qquad b^2+c^2\geq 2bc,\qquad c^2+a^2\geq 2ca.$$

Adding these three inequalities and dividing by 2 gives the desired $a^2+b^2+c^2\geq ab+bc+ca$. (Or, simply rearrange $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$).

(b) Just like above, the expression can be derived by adding AM-GM inequality applied on each pair of variables, or simply by expanding and rearranging the clearly true inequality

$$\sum_{\text{sym}} (a-b)^2 \geq 0.$$

("sym" means the sum of the term over all combinations (in this case, pairs) of the variables a, b, c, d). From the above expression, we can see that equality would occur when $a = b = c = d$.

Solution 8

Solution 1: We have $x = 12 - \frac{12y}{5}$, so then plugging this in to the expression $x^2 + y^2$ gives

$$x^2 + y^2 = \frac{169y^2}{25} - \frac{288y}{5} + 144.$$

Completing the square on this expression gives

$$\left(\frac{13y}{5} - \frac{144}{13}\right)^2 + 144 - \frac{144^2}{13^2}.$$

Thus the answer is

$$\sqrt{144 - \frac{144^2}{13^2}} = \boxed{\frac{60}{13}}.$$

Solution 2: If we view $\sqrt{x^2+y^2}$ as the distance from a point (x, y) to the origin, then we see that the answer is the minimal distance from $(0, 0)$ to the line $5x + 12y = 60$. This is also the altitude to the hypotenuse in a 5-12-13 triangle, which can be computed with similar triangles to be $\frac{60}{13}$, as found above.

Solution 9

(a) We have that $x^2 + 16 = xy$. But by AM-GM,

$$2\sqrt{(x^2)(16)} \leq x^2 + 16 = xy \implies 8x \leq xy.$$

Thus $y \geq \boxed{8}$.

(b) Note that $\frac{y}{x}$ is the slope of the line from the origin to a point (x, y). Since this is an ellipse in the first quadrant, it is then clear that this slope should take its maximum and minimum values at the tangent lines to the ellipse from the origin (draw a picture to see this).

To find the slopes of these lines: let $m = \frac{y}{x} \implies y = mx$. Then, depending on the value of m, there should be either $0, 1$, or 2 solutions to the equation of the ellipse, based on how many times the line $y = mx$ intersects the ellipse. We want to find m which give only one solution. Substituting $y = mx$ into the equation gives

$$2x^2 + x(mx) + 3(mx)^2 - 11x - 20(mx) + 40 = 0 \implies x^2(3m^2 + m + 2) + x(-20m - 11) + 40 = 0.$$

In order for there to be one solution, the discriminant must be 0; that is,

$$(-20m - 11)^2 - 4(3m^2 + m + 2)(40) = 0 \implies -80m^2 + 280m - 199 = 0.$$

We want the sum of the m values which solve this equation, which is $\frac{280}{80}$ by Vieta's formulas, so the answer is $\boxed{\textbf{(C)}}$.

Solution 10

We complete the square in the numerator and apply AM-GM:

$$\frac{(x+1)^2+4}{x+1} \geq \frac{2\sqrt{4(x+1)^2}}{x+1} = \boxed{4}.$$

Equality holds when $(x+1)^2 = 4 \implies x = 2$.

Solution 11

For each $k \geq 2$, note that

$$\begin{aligned}(1+a_k)^k &= \Bigg(\underbrace{\frac{1}{k-1}+\cdots+\frac{1}{k-1}}_{k-1\text{ terms}}+a_k\Bigg)^k \\ &\geq k^k\frac{a_k}{(k-1)^{k-1}}. && \text{by AM-GM}\end{aligned}$$

Thus, we have

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n \geq \frac{2^2}{1^1}\cdot\frac{3^3}{2^2}\cdots\frac{n^n}{(n-1)^{n-1}}\cdot(a_2a_3\cdots a_n) = n^n,$$

upon cancelling the fractions and substituting $a_2a_3\cdots a_n = 1$. But also, note that equality only holds when $a_k = \dfrac{1}{k-1}$ for each k; so, since $a_2a_3\cdots a_n = 1$, equality cannot hold. Thus we actually have strict inequlity, as desired. □

Solution 12

$$\begin{aligned}a^2bc+b^2cd+c^2da+d^2ab &= ac(ab+cd)+bd(bc+ad) \\ &= ac(ab+cd)+ac(ac+bd)+bd(bc+ad)+bd(ac+bd)-(ac+bd)^2\end{aligned}$$