ORMC Olympiad Group
Week 4
Inequalities I: AM-GM

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Problems

1. Let x, y be real numbers. What is the minumum value of $x^2+y^2+4x-6y$?

2. (a) Let $x > 0$ be a real number. Find the smallest value that $\frac{x^2+1}{x}$ can take. For which x equality occurs?

   (b) Let $x > 0$ be a real number. Find the smallest value that $\frac{12x^2+3}{x}$ can take. For which x equality occurs?

   (c) Let $x, y > 0$ be real numbers. Find the smallest value that $\frac{x^4+y^4+162}{xy}$ can take. For which x and y the equality occurs?
   HINT: Use $AM-GM$ for 4 numbers: $a^4+b^4+c^4+d^4 \geq 4abcd$.

   (d) Let $x, y > 0$ be real numbers. Find the smallest value that $\frac{x^4+y^4+1}{xy}$ can take. For which x and y the equality occurs?

3. **(HMMT 2005 Guts)** Let x, y, and z be positive real numbers such that $xy + z = (x + z)(y + z)$. What is the maximum possible value of xyz?

4. **(AIME 1983)** Find the minimum value of $\frac{9x^2 \sin^2 x+4}{x \sin x}$ for $0 < x < \pi$.

5. For $0 \leq a \leq 6$, find the maximum value of $a^2(12-2a)$.

6. (a) Find the maximum value of $3x - x^3$

   (b) Find the maximum value of $2x - x^3$.

7. (a) Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$.

   (b) Prove that for any reals a, b, c, d, the inequality

$$a^2 + b^2 + c^2 + d^2 \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

   holds. When does the equality occur?

8. **(AHSME 1961)** Find the minimum value of $\sqrt{x^2 + y^2}$ if $5x+12y = 60$.

9. (a) x, y are two positive real numbers satisfying $x^2 - xy = -16$. Find the minimum possible value of y.

   (b) **(AMC12-2004B)** The graph of $2x^2+xy+3y^2-11x-20y+40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of $a + b$?

   (A) 3 (B) $\sqrt{10}$ (C) $\frac{7}{2}$ (D) $\frac{9}{2}$ (E) $2\sqrt{14}$

10. Assume x is a positive real number. Find the minimal value of $\frac{x^2+2x+5}{x+1}$.

11. **(IMO 2012 Problem 2)** Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

12. a, b, c, d are positive real numbers and $a + b + c + d = 4$. Show that

$$\sum_{cyc} a^2 bc \leq 4$$