ORMC Olympiad Group
Week 3

Osman Akar

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Problems

1. Find formulas for the following recursive functions

   (a) $x_1 = 7, x_2 = 25$, and for $n \geq 1$

   $$x_{n+2} = 7x_{n+1} - 12x_n$$

   (b) $x_1 = 1, x_2 = 5$, and for $n \geq 1$

   $$x_{n+2} = x_{n+1} + 2x_n$$

   (c) $x_1 = 7, x_2 = 23$, and for $n \geq 1$

   $$x_{n+2} = x_{n+1} + 2x_n$$

   (d) $x_1 = 3, x_2 = 10$, and for $n \geq 1$

   $$x_{n+2} = 4x_{n+1} + 12x_n$$

   (e) $x_1 = 4, x_2 = 10$, and for $n \geq 1$

   $$x_{n+2} = 4x_{n+1} - 4x_n$$

(f) Let $x_1 = 0, x_2 = 3, x_3 = 11$ and for $n \geq 1$

$$x_{n+3} = 6x_{n+2} - 11x_{n+1} + 6x_n$$

2. **(TJNMO)** Integers $x_0, x_1, \dots x_{2018}$ satisfy $x_0 = 1$, $x_1 = 2$ and $x_{n+1} = 3x_n - 2x_{n-1}$ for all $n \geq 1$. Find the remainder when x_{2018} divided by 2018.

3. Assuming $x + \frac{1}{x}$ is an integer, prove that $x^n + \frac{1}{x^n}$ is also integer for all integers n.

4. Let α be the real value so that $\alpha + \frac{1}{\alpha} = 4$. Compute $\alpha^5 + \frac{1}{\alpha^5}$.

5. Find $\lfloor (\frac{\sqrt{5}+1}{2})^9 \rfloor$

6. a, b, c are complex numbers such that $a + b + c = 1$, $a^2 + b^2 + c^2 = 2, a^3 + b^3 + c^3 = 3$.

   (i) What is $a^4 + b^4 + c^4$?

   (ii) What is $a^5 + b^5 + c^5$?

7. **(The USSR Olympiad Problem Book 4.106)** Does there exist integer n such that $\{(2 + \sqrt{2})^n\} > 0.999999999999$? Either show existence, or prove that there is no such n. Here, by $\{x\}$ we represent the fractional part of x. I.e. $\{x\} = x - \lfloor x \rfloor$.

8. **(HMMT Guts 2010F)** Define a sequence of polynomials as follows: Let $a_1 = 3x^2 - x$, let $a_2 = 3x^2 - 7x + 3$, and for $n \geq 1$, let $a_{n+2} = \frac{5}{2}a_{n+1} - a_n$. As n tends to infinity, what is the limit of the sum of the (complex or real) roots of a_n?

9. **(TNMO-FR 2018)** Let $x_0 = 2018$, and for $n \geq 1$ $x_n = x_{n-1} - 12$ or $x_n = 9x_{n-1} - 4$. For which of the fallowing values could be x_n for some n?

   (A) 100 **(B)** 10^{100} **(C)** 2018^{100} **(D)** $2018^{2018} - 2018$ (E) None

10. **(AOPS Mock AMC 2017 Summer)** The Tribonacci sequence is defined as $T_1 = 1, T_2 = 1, T_3 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n > 3$. What is the remainder when T_{2017} is divided by 12?

11. **(Putnam 2018)** Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.