

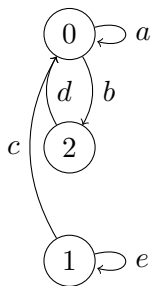
FALL 2021

OLGA RADKO MATH CIRCLE
ADVANCED 2
NOVEMBER 7TH, 2021

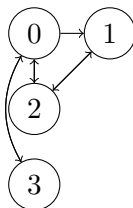
Each problem is worth 2 points unless otherwise noted.

1. COMBINATORICS ON WORDS

Problem 1.1. Find the line graph associated to the following directed graph.



Problem 1.2. Find all Eulerian paths of the following graph starting at vertex 0.



Problem 1.3. Find a de Bruijn word of order 3 in a binary alphabet.

Problem 1.4. Find a Sturmian word of order 4 in a binary alphabet.

Problem 1.5. What is the length of a de Bruijn word in a ternary alphabet?

Problem 1.6. How many distinct de Bruijn words of order 2 are there in a binary alphabet?

2. NIMBERS

For reference, the Nimber addition table for 0 through 7 is given below.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Problem 2.1. Which player wins the following Nim games? Answer all correctly for points (no partial credit).

- (1) $*5 + *6 + *7$
- (2) $*12 + *12$
- (3) $*1 + *2 + *3 + *4 + *5 + *6 + *7$

Problem 2.2. Which player wins the Nim game $*28 + *5 + *25$?

Problem 2.3. Play a game of $*5 + *6 + *7$ Nim with an instructor. Go first. Win the game for 2 points.

Problem 2.4. Play a game of $*5 + *7 + *7$ Nim with an instructor. Go second. The instructor will make a mistake at some point in the game. By punishing the mistake, win the game for 2 points.

Problem 2.5. Consider the following game: There is a single stack of n coins. Players take turns removing either 2 coins or 3 coins from the stack. The player who cannot make a legal move loses. Determine all values of n for which the first player has a winning strategy.

Problem 2.6. The instructor will choose ten vectors on \mathbb{R}^2 (e.g., $(3, -1)$, $(2, 2)$). The instructor and you take turns picking vectors. After all vectors are picked, both of you will compute the sum of the length of your vectors (the length of a vector (a, b) is $\sqrt{a^2 + b^2}$). The person with the larger value wins. Play this game twice with an instructor, once as the first player, once as the second player.

3. MISCELLANEOUS

Problem 3.1. Choose a game of chance and play it twice with an instructor. You gain the full points of this question if you win one of the two games, and 0 points otherwise. The game you choose must be a fair game (i.e., both you and the instructor have 50 percent chance of winning), and the instructor reserves the right to refuse to play if you cannot convince him/her that your game is fair.

Problem 3.2. How many ways can 2021 be written as the difference between squares of two integers?

Problem 3.3. Find a perfect square which consists of 4 digits such that first 2 of them are the same and last 2 are the same.

Problem 3.4. Find the coefficients of x^{17} and x^{18} in $(1 + x^5 + x^7)^{20}$.

Problem 3.5. Determine the last two digits of 7^{777777} .

Hint: ask your instructor about Fermat's little Theorem, if you do not know it already.

Problem 3.6. Let n be a positive integer; let $x(n) \geq n$ be the smallest integer such that among any $x(n)$ number of integers, one can find n of them whose sum is divisible by n . Find $x(1024)$.

Problem 3.7. Consider the polynomial $f(x) = x^9 + 2x^8 + 3x^7 + 4x^6 + 5x^5 + 6x^4 + 7x^3 + 8x^2 + 9x + q$ (where q is some variable that you don't know the value of). Being a 9-th degree polynomial, there are 9 such values of x such that $f(x) = 0$. Find the sum of all such values (the roots of the polynomial).

Problem 3.8. (a) What is the smallest number of rooks which can be arranged on an 8×8 chessboard in such a way that every square of the board is controlled by at least one of them?
(b) In how many different ways can this be done?

Problem 3.9. Compute the product

$$\cos \frac{\pi}{4} \times \cos \frac{\pi}{8} \times \cos \frac{\pi}{16} \times \dots \times \cos \frac{\pi}{2^n} \times \dots$$

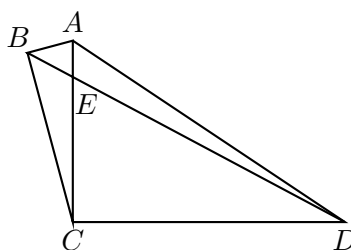
Problem 3.10. Every minute a cell has equal chance of dying, staying unchanged or dividing into 2. Starting with 1 cell, what is the probability that the cell would eventually go extinct at some point?

Problem 3.11. Find the last two digits of

$$\text{lcm}(5^{1!} + 1, 5^{2!} + 1, 5^{3!} + 1, \dots, 5^{2021!} + 1).$$

Problem 3.12. If a, b, c are rational roots of $x^3 + mx + n$ with $a \neq 0$, find all rational roots of $ax^2 + bx + c$.

Problem 3.13. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $AD = 30$. Diagonals AC and BD intersect at E and $AE = 5$. Find the area of $ABCD$. The below diagram is not drawn to scale.



Problem 3.14. If P is a two-dimensional plane in \mathbb{R}^3 and x is a point in \mathbb{R}^3 , the projection of x on P is the unique point y on P such that the segment xy is perpendicular to P . Suppose now that A is a subset of \mathbb{R}^3 , and there are two planes such that the projection of A onto them are both (two-dimensional) disks. Show that these two disks must have equal radius.

Problem 3.15. Let \mathbb{Z} be the set of integers. Determine all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all integers a and b , we have

$$f(2a) + 2f(b) = f(f(a + b)).$$

Hint: Try to plug in $a = 0$ and $a = 1$ into the relation above and get two relations; link the two relations you got.

Problem 3.16. Find the upper bound on z if

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2} z^k$$

converges.

Problem 3.17. What is the smallest whole number that, when written out, uses all the vowels (aeiouy) exactly once each in its formal English spelling?