Olga Radko Math Circle Competition

Advanced 3

Week 5 Fall 2021

Each problem is worth 2 points unless otherwise noted.

1. Nimbers

Problem 1.1. Let's play Nim! Get some number of players and ask Aaron. You will go to a separate breakout room, and the game will be played over jamboard.

Problem 1.2. Consider playing Nim with a single pile but you are only allowed to remove a number of stones congruent to 1 or 6 mod 8. What are the losing positions?

Problem 1.3. Consider playing Nim with a single pile but you are only allowed to remove 1, 2, or 5 stones. What are the losing positions?

Problem 1.4. Fix a finite set S of positive integers. Consider playing Nim with a single pile but you are only allowed to remove a number stones that is in S. Are the losing positions always periodic?

2. FFT

Problem 2.1. For each pair f, g determine if $f \in O(g)$.
Each part is worth 1 point, based only on whether the answer is correct. Only one attempt will be allowed. All parts must be attempted together.
a: $f(x) = x^2$, $g(x) = x$
b: $f(x) = x^{0.001}$, $g(x) = \log x$.
c: $f(x) = x^{\frac{1}{1+\log x}}$, $\log x$
d: $f(x) = \frac{1}{e^{\frac{1}{x}}-1}$, $g(x) = x$.

Problem 2.2. Compute

$$\sum_{j=0}^{\infty} \frac{\cos(j\theta)}{2^j}$$

where $\theta = \arccos(\frac{1}{5})$.

Problem 2.3. What is the product of all the 50-th roots of unity. What about the 2021-th roots of unity?

3. Miscellaneous Problems

Problem 3.1. Show there are no integers x, y so that $x^2 + 3xy - 2y^2 = 122$.
Hint: mod 17.

Problem 3.2. There is a knight on each square of a 2021 by 2021 chess board. Is there a way to move each knight simultaneously so that no two knights end up on the same square?

Problem 3.3. Suppose there are $2n$ points in the plane with no three points collinear. Half are colored red and half are colored blue. Is it always possible to draw n line segments so that no two line segments cross and each line segment connects a red point to a blue point?

Problem 3.4. Alice the science teacher ordered a new weight set. However she is worried that the labels on her weights might have gotten mixed up. She has a weight that she knows is exactly one gram and a fulcrum with distance markings. Unfortunately the fulcrum is very cheap and will break after one use. How can Alice determine if the labels on her weight set are mixed up?

Problem 3.5. Bob and Carol play a betting game. Carol shuffles a deck and deals cards face up one at a time. Bob can interrupt at any time at which point he bets \$1 at even odds that the next card dealt will be red. He must interrupt before the last card is dealt and he may only interrupt once. What is his expected winnings with the optimal strategy?