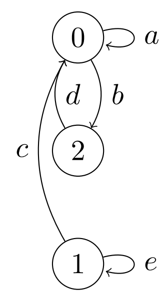
# FALL 2021

OLGA RADKO MATH CIRCLE
ADVANCED 2
NOVEMBER 7TH, 2021

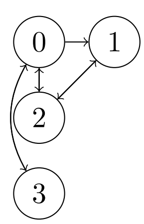
Each problem is worth 2 points unless otherwise noted.

## 1. Combinatorics on words

**Problem 1.1.** Find the line graph associated to the following directed graph.




**Problem 1.2.** Find all Eulerian paths of the following graph starting at vertex 0.




**Problem 1.3.** Find a de Bruijn word of order 3 in a binary alphabet.

**Problem 1.4.** Find a Sturmian word of order 4 in a binary alphabet.

**Problem 1.5.** What is the length of a de Bruijn word in a ternary alphabet?

**Problem 1.6.** How many distinct de Bruijn words of order 2 are there in a binary alphabet?

## 2. Nimbers

For reference, the Nimber addition table for 0 through 7 is given below.

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 2 | 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 3 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 6 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

**Problem 2.1.** Which player wins the following Nim games? Answer all correctly for points (no partial credit).

(1) $*5 + *6 + *7$
(2) $*12 + *12$
(3) $*1 + *2 + *3 + *4 + *5 + *6 + *7$

**Problem 2.2.** Which player wins the Nim game $*28 + *5 + *25$?

**Problem 2.3.** Play a game of $*5 + *6 + *7$ Nim with an instructor. Go first. Win the game for 2 points.

**Problem 2.4.** Play a game of $*5+*7+*7$ Nim with an instructor. Go second. The instructor will make a mistake at some point in the game. By punishing the mistake, win the game for 2 points.

**Problem 2.5.** Consider the following game: There is a single stack of $n$ coins. Players take turns removing either 2 coins or 3 coins from the stack. The player who cannot make a legal move loses. Determine all values of $n$ for which the first player has a winning strategy.

**Problem 2.6.** The instructor will choose ten vectors on $\mathbb{R}^2$ (e.g., $(3, -1)$, $(2, 2)$). The instructor and you take turns picking vectors. After all vectors are picked, both of you will compute the sum of the length of your vectors (the length of a vector $(a, b)$ is $\sqrt{a^2 + b^2}$). The person with the larger value wins. Play this game twice with an instructor, once as the first player, once as the second player.

## 3. Miscellaneous

**Problem 3.1.** Choose a game of chance and play it twice with an instructor. You gain the full points of this question if you win one of the two games, and 0 points otherwise. The game you choose must be a fair game (i.e., both you and the instructor have 50 percent chance of winning), and the instructor reserves the right to refuse to play if you cannot convince him/her that your game is fair.

**Problem 3.2.** How many ways can 2021 be written as the difference between squares of two integers?

**Problem 3.3.** Find a perfect square which consists of 4 digits such that first 2 of them are the same and last 2 are the same.

**Problem 3.4.** Find the coefficients of $x^{17}$ and $x^{18}$ in $(1 + x^5 + x^7)^{20}$.

**Problem 3.5.** Determine the last two digits of $7^{777777}$.
Hint: ask your instructor about Fermat's little Theorem, if you do not know it already.

**Problem 3.6.** Let $n$ be a positive integer; let $x(n) \geq n$ be the smallest integer such that among any $x(n)$ number of integers, one can find $n$ of them whose sum is divisible by $n$. Find $x(1024)$.

**Problem 3.7.** Consider the polynomial $f(x) = x^9 + 2x^8 + 3x^7 + 4x^6 + 5x^5 + 6x^4 + 7x^3 + 8x^2 + 9x + q$ (where $q$ is some variable that you don't know the value of). Being a 9-th degree polynomial, there are 9 such values of $x$ such that $f(x) = 0$. Find the sum of all such values (the roots of the polynomial).

**Problem 3.8.** (a) What is the smallest number of rooks which can be arranged on an $8 \times 8$ chessboard in such a way that every square of the board is controlled by at least one of them?
(b) In how many different ways can this be done?

**Problem 3.9.** Compute the product

$$\cos\frac{\pi}{4} \times \cos\frac{\pi}{8} \times \cos\frac{\pi}{16} \times \ldots \cos\frac{\pi}{2^n} \times \ldots$$

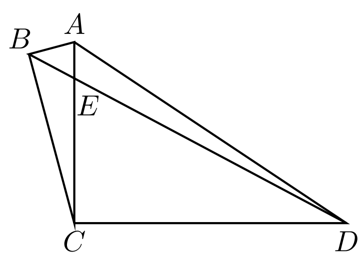
**Problem 3.10.** Every minute a cell has equal chance of dying, staying unchanged or dividing into 2. Starting with 1 cell, what is the probability that the cell would eventually go extinct at some point?

**Problem 3.11.** Find the last two digits of

$$\operatorname{lcm}(5^{1!} + 1, 5^{2!} + 1, 5^{3!} + 1, \cdots, 5^{2021!} + 1).$$

**Problem 3.12.** If $a, b, c$ are rational roots of $x^3 + mx + n$ with $a \neq 0$, find all rational roots of $ax^2 + bx + c$.

**Problem 3.13.** Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90°$, $AC = 20$, and $AD = 30$. Diagonals $AC$ and $BD$ intersect at $E$ and $AE = 5$. Find the area of $ABCD$. The below diagram is not drawn to scale.




**Problem 3.14.** If $P$ is a two-dimensional plane in $\mathbb{R}^3$ and $x$ is a point in $\mathbb{R}^3$, the projection of $x$ on $P$ is the unique point $y$ on $P$ such that the segment $xy$ is perpendicular to $P$. Suppose now that $A$ is a subset of $\mathbb{R}^3$, and there are two planes such that the projection of $A$ onto them are both (two-dimensional) disks. Show that these two disks must have equal radius.

**Problem 3.15.** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers $a$ and $b$, we have

$$f(2a) + 2f(b) = f(f(a + b)).$$

*Hint:* Try to plug in $a = 0$ and $a = 1$ into the relation above and get two relations; link the two relations you got.

**Problem 3.16.** Find the upper bound on $z$ if

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2} z^k$$

converges.

**Problem 3.17.** What is the smallest whole number that, when written out, uses all the vowels (aeiouy) exactly once each in its formal English spelling?