Week 6: Frieze patterns 1

Nikita

Problem 1.
$a_1 = 1$, $a_2 = 1000$, $a_{n+1} = a_n - a_{n-1}$. Find a_{2021}.

Problem 2.
$a_1 = 1$, $a_2 = 1000$, $a_{n+1} = \frac{1+a_n}{a_{n-1}}$. What is a_{2021}?

Definition 1.
Coxeter's **frieze patterns** are arrays of numbers satisfying the following properties:

1. the array has finitely many rows, all of them being infinite on the right and left,
2. the first two top rows are a row of 0's followed by a row of 1's, and the last two bottom rows are a row of 1's followed by a row of 0's
3. consecutive rows are displayed with a shift, and every four adjacent entries a, b, c, d forming a diamond

$$\begin{array}{ccc} & \text{b} & \\ \text{a} & & \text{d} \\ & \text{c} & \end{array}$$

   satisfy the unimodular rule: $ad - bc = 1$.

Example 1.
Frieze of width 5

$$\begin{array}{cccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Problem 3.
Continue the following frieze pattern:

$$\begin{array}{cccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & ? & & ? & & ? & & ? & & ? & & \cdots \\
\cdots & & 1000 & & ? & & ? & & ? & & ? & & ? & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Problem 4.
Continue the following frieze pattern. What is its width (the number of rows between rows of 1)?

0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
4		2		1		3		2		2		1		4		2	
	?		?		?		?		?		?		?		?		?
?		?		?		?		?		?		?		?		?	
	?		?		?		?		?		?		?		?		?

Problem 5.
Continue the following infinite frieze pattern.

0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
2		2		2		2		2		2		2		2		2	
	?		?		?		?		?		?		?		?		?
?		?		?		?		?		?		?		?		?	
	?		?		?		?		?		?		?		?		?

Problem 6.
Continue the following infinite frieze pattern. Prove that all the numbers one obtains will be positive integers.

0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
3		3		3		3		3		3		3		3		3	
	?		?		?		?		?		?		?		?		?
?		?		?		?		?		?		?		?		?	
	?		?		?		?		?		?		?		?		?
?		?		?		?		?		?		?		?		?	

Problem 7.
Continue the following finite frieze pattern, where $\phi = \frac{\sqrt{5}+1}{2}$

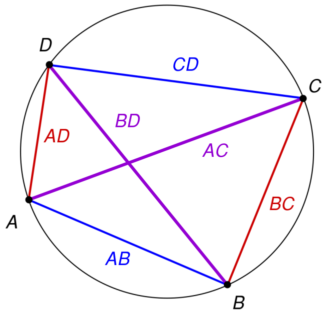
0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
ϕ		ϕ		ϕ		ϕ		ϕ		ϕ		ϕ		ϕ		ϕ	
	?		?		?		?		?		?		?		?		?
?		?		?		?		?		?		?		?		?	
	?		?		?		?		?		?		?		?		?

The pattern you see here can be generalized.

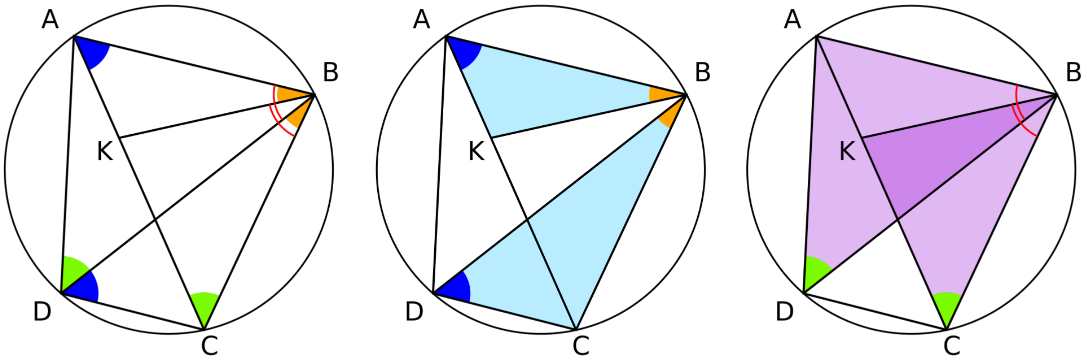
Problem 8.
Consider a regular pentagon $ABCDE$ with side length 1. Use angle chasing, equal and similar triangles to prove that its diagonal is equal to ϕ.

Theorem 1 (Ptolemy)**.**
If $ABCD$ is inscribed in circle,

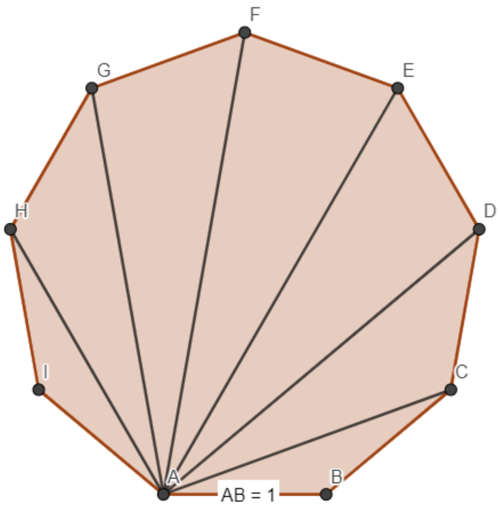
$$|\overline{AC}| \cdot |\overline{BD}| = |\overline{AB}| \cdot |\overline{CD}| + |\overline{BC}| \cdot |\overline{AD}|$$



Problem 9.
Prove Ptolemy's theorem with the help of the following picture.



Problem 10.
Consider the regular 9-gon $ABCDEFGHI$ with side length 1.



Prove that it produces a finite frieze

0 0 0 0 0 0 0 0

AB AB AB AB AB AB AB

AC AC AC AC AC AC AC AC

AD AD AD AD AD AD AD

AE AE AE AE AE AE AE AE

AF AF AF AF AF AF AF

AG AG AG AG AG AG AG AG

AH AH AH AH AH AH AH

AI AI AI AI AI AI AI AI

0 0 0 0 0 0 0