

Week 6: Frieze patterns 1

Nikita

Problem 1.

$a_1 = 1, a_2 = 1000, a_{n+1} = a_n - a_{n-1}$. Find a_{2021} .

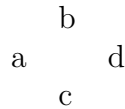
Problem 2.

$a_1 = 1, a_2 = 1000, a_{n+1} = \frac{1+a_n}{a_{n-1}}$. What is a_{2021} ?

Definition 1.

Coxeter's **frieze patterns** are arrays of numbers satisfying the following properties:

1. the array has finitely many rows, all of them being infinite on the right and left,
2. the first two top rows are a row of 0's followed by a row of 1's, and the last two bottom rows are a row of 1's followed by a row of 0's
3. consecutive rows are displayed with a shift, and every four adjacent entries a, b, c, d forming a diamond



satisfy the unimodular rule: $ad - bc = 1$.

Example 1.

Frieze of width 2

	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1
	1	2	2	1	3	1	...	
...	1	3	1	2	2	1	...	
	1	1	1	1	1	1	1	...
...	0	0	0	0	0	0	0	...

Problem 3.

Continue the following frieze pattern:

	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1
	1	?	?	?	?	?	?	...
...	1000	?	?	?	?	?	?	...
	1	1	1	1	1	1	1	...
...	0	0	0	0	0	0	0	...

Problem 4.

Continue the following frieze pattern. What is its width (the number of rows between rows of 1)?

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
4	2	1	3	2	2	1	4	2	
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?

Problem 5.

Continue the following infinite frieze pattern.

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?

Problem 6.

Continue the following infinite frieze pattern. Prove that all the numbers one obtains will be positive integers.

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
3	3	3	3	3	3	3	3	3	3
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?

Problem 7.

Continue the following finite frieze pattern, where $\phi = \frac{\sqrt{5}+1}{2}$

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?
?	?	?	?	?	?	?	?	?	?

The pattern you see here can be generalized.

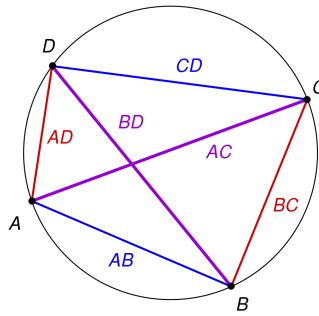
Problem 8.

Consider a regular pentagon $ABCDE$ with side length 1. Use angle chasing, equal and similar triangles to prove that its diagonal is equal to ϕ .

Theorem 1 (Ptolemy).

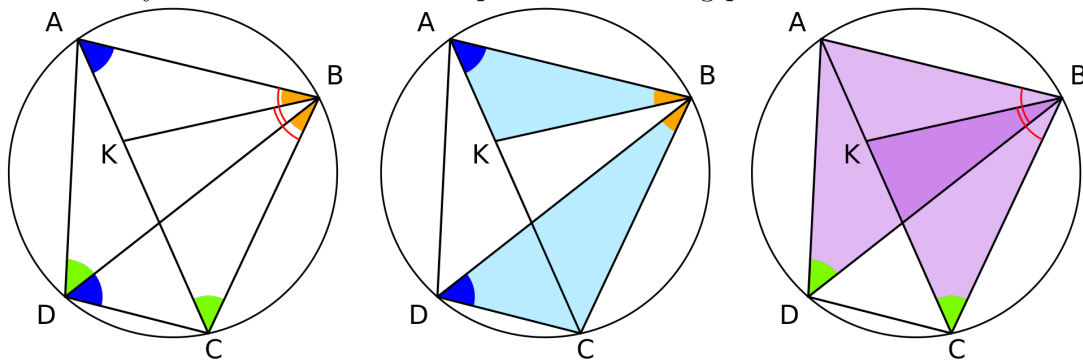
If $ABCD$ is inscribed in circle,

$$|\overline{AC}| \cdot |\overline{BD}| = |\overline{AB}| \cdot |\overline{CD}| + |\overline{BC}| \cdot |\overline{AD}|$$



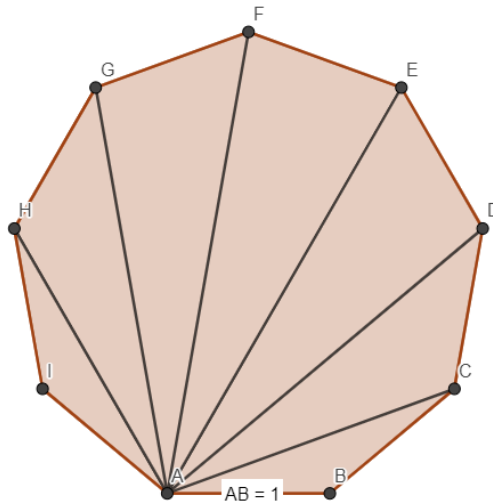
Problem 9.

Prove Ptolemy's theorem with the help of the following picture.



Problem 10.

Consider the regular 9-gon $ABCDEFGHI$ with side length 1.



Prove that it produces a finite frieze:

0	0	0	0	0	0	0	0	0	0	0	0
<i>AC</i>	<i>AB</i>	<i>AC</i>	<i>AB</i>	<i>AC</i>	<i>AB</i>	<i>AC</i>	<i>AB</i>	<i>AC</i>	<i>AB</i>	<i>AC</i>	<i>AB</i>
<i>AE</i>	<i>AD</i>	<i>AE</i>	<i>AD</i>	<i>AE</i>	<i>AD</i>	<i>AE</i>	<i>AD</i>	<i>AE</i>	<i>AD</i>	<i>AE</i>	<i>AD</i>
<i>AG</i>	<i>AF</i>	<i>AG</i>	<i>AF</i>	<i>AG</i>	<i>AF</i>	<i>AG</i>	<i>AF</i>	<i>AG</i>	<i>AF</i>	<i>AG</i>	<i>AF</i>
<i>AI</i>	<i>AH</i>	<i>AI</i>	<i>AH</i>	<i>AI</i>	<i>AH</i>	<i>AI</i>	<i>AH</i>	<i>AI</i>	<i>AH</i>	<i>AI</i>	<i>AH</i>
0	0	0	0	0	0	0	0	0	0	0	0