Math Circle Advanced 1

Game of Halloween 2021

1 Algebra

Problem 1.1.
Find all real solutions (x, y, z) to the equation $x^2 + 4y^2 + 9z^2 + 16 = 2x + 12y - 12z + 2$.

Problem 1.2.
Find the coefficients of x^{17} and x^{18} in $(1 + x^5 + x^7)^{20}$.

Problem 1.3.
Let $f(x, y, z)$ be a function given by

$$f(x, y, z) \; := \; \frac{x^2 + y^2 + z^2}{xy + xz + yz}$$

where x, y, z are positive real numbers. Find the minimum value of f.

Problem 1.4.
Find the value of the product

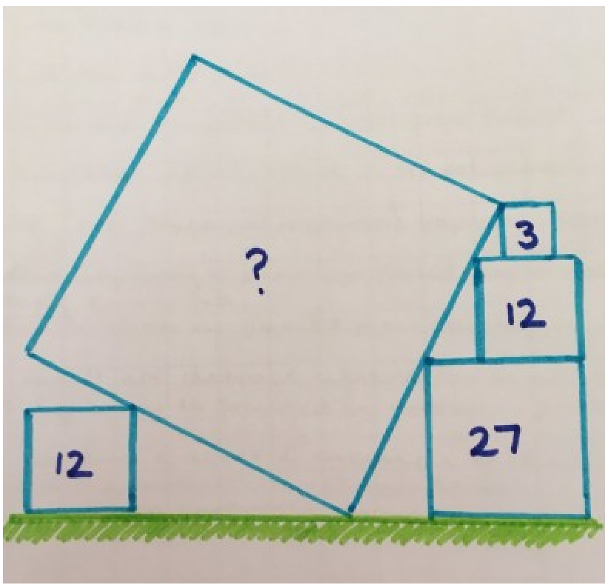
$$(1 - \frac{1}{4})(1 - \frac{1}{9}) \dots (1 - \frac{1}{100}),$$

where the numbers in the denominators are equal to the squares of natural numbers from 2 to 10.

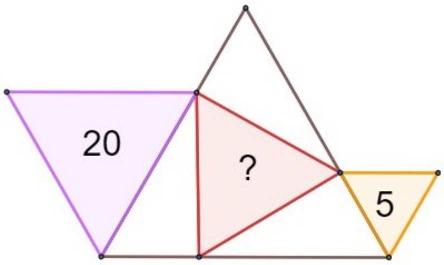
Problem 1.5.
How many ways can 2021 be written as the difference between squares of two integers?

2 Geometry

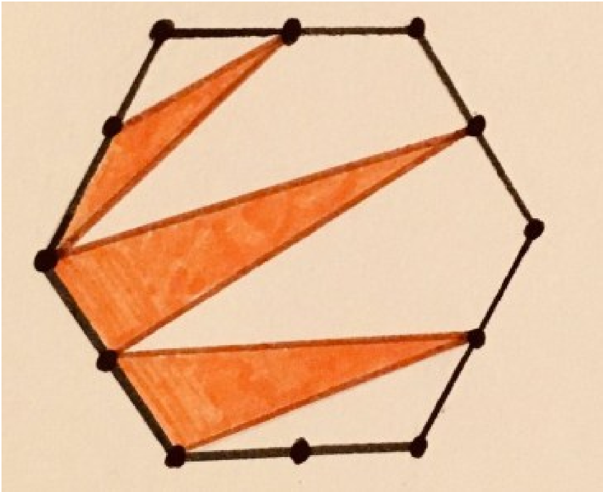
Problem 2.1.
Areas of some squares are given. Find the area of the toppled square.



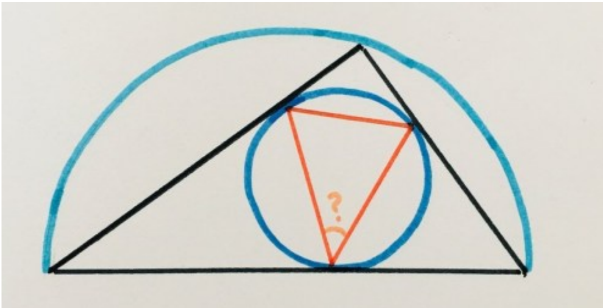
Problem 2.2.
All Men are Created Equilateral. Find the area



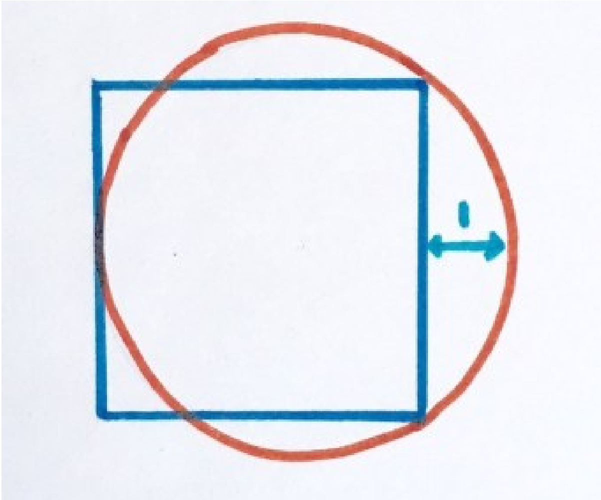
Problem 2.3.
What fraction is shaded? The hexagon is regular, with equally spaced dots around its perimeter.



Problem 2.4.
Koschei hides this angle in a triangle inside a circle inside a triangle inside a semicircle. Find the degree of this angle!



Problem 2.5.
What's the area of the square?



3 Number Theory

Problem 3.1.
Find all integer solution (x, y) to the equation $x^2 + y^2 = 81$.

Problem 3.2.
Find a perfect square which consists of 4 digits such that first 2 of them are the same and last 2 are the same.

Problem 3.3.
Find the square root of 102030405060708090807060504030201, and explain your solution.

Problem 3.4.
A wizard wrote each number from 1 to 10^{10} in formal English (e.g., "three hundred forty five", "one thousand seventy two"), and then listed them in the alphabetical order (as in a dictionary, where spaces and hyphens are ignored). What is the first odd number in the list?

Problem 3.5.
Let $k = 2021^3 + 3^{2021}$. What is the units digit of $k^3 + 3^k$?

4 Combinatorics

Problem 4.1.
Find the number of non-negative solutions (x, y, z) such that $x + y + z = 10$. Here x, y, z are all non-negative.

Problem 4.2.
We introduce a new chess piece that we call the zombie-piece. The zombie-piece starts at the bottom left of the chessboard, and at every step the zombie-piece can either move one step up or one step right. How many ways can the zombie-piece move from the bottom left to the top right of the chessboard?

Problem 4.3.
(a) What is the smallest number of rooks which can be arranged on an 8×8 chessboard in such a way that every square of the board is controlled by at least one of them?
(b) In how many different ways can this be done?

Problem 4.4.
You (the hero) will play a counting game against an instructor (a vampire). The hero and the vampire take turns in choosing a number from from $\{1, 2, 3\}$ (a number can be chosen more than once). If at some point, the sum of all chosen numbers of a player add up to or exceeding 21, then the player loses. This game will be played twice, once with the hero starts first, and once with the hero starts second. Each round of the game is worth half the points of this question.

Problem 4.5.
You (the hero) will play the following game against an instructor (the sorcerer). On a table there is a row of these ten numbers:

$$9,\ 10,\ 5,\ 6,\ 3,\ 13,\ 12,\ 5,\ 9,\ 5.$$

The hero (you) picks a number from one of the ends and keep it in their hand; then the sorcerer (instructor) chooses a number from one of the remaining ends, and the alternation continues until the sorcerer picks the last number.

The hero wins if they have at least the same sum as the sorcerer at the end. The hero gets the the full points for this question for winning, and 0 points otherwise.

5 Probability

Problem 5.1.
Choose a game of chance and play it with an instructor. You gain the full points of this question if you win the game, and 0 points otherwise. The game you choose must be a fair game (i.e., both you and the instructor have 50 percent chance of winning), and the instructor reserves the right to refuse to play if you cannot convince him/her that your game is fair.

Problem 5.2.
Suppose that a boy remembers all but the last figure of his home's telephone number and decides to choose the last figure at random in an attempt to call his home. If he has only two quarters in his pocket (one call costs one quarter), what is the probability that he will dial the right number before he runs out of money ?

Problem 5.3.
Choose two numbers randomly, not necessarily distinct, from the set $\{1, 2, 3, 4, 5, 6\}$. What is the probability that their sum is 10?

Problem 5.4.
A vampire and a werewolf plan to meet some time between 9am and 10am on Sunday in front of the UCLA student store. They agree on the plan that anyone who arrives will wait for 5 minutes before 10am and leave if not seeing the other person within that waiting time.

What is the probability that they successfully meet each other between 9am and 10am?

Problem 5.5.
Ten vampires line up to board an airplane to Castlevania, but the first vampire lost his boarding pass and takes a random seat instead. Each subsequent vampire takes their assigned seat if available, or take a random unoccupied seat otherwise.

What is the probability that the last vampire to board ends up sitting in her own chair?