Cryptarithms

Cryptarithmetic, also know as cryptarithm, alphametics, or word addition, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Problem 1 *Solve the following cryptarithm.*

$$\begin{array}{cccc} & & H & E \\ + & & H & E \\ & & H & E \\ \hline & S & H & E \end{array}$$

The following cryptarithm[1] is a bit more complicated.

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

Let's try to solve it together. First, $S + M \geq 10$ because the result carries over from the fourth column to the fifth. A sum of single digit numbers cannot exceed 18, so $M = 1$.

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & 1 & O & R & E \\ \hline & 1 & O & N & E & Y \end{array}$$

Now, $S + 1 \geq 10$. This can only happen if either $S = 9$ or $S = 8$. Let us consider the latter case. If $S = 8$, then $O = 0$ and one is carried over from the third column, implying $E = 9$ and $N = 0$. But, since $O = 0$, $N \neq 0$. Thus, $S = 9$.

$$\begin{array}{cccccc} & & 9 & E & N & D \\ + & & 1 & O & R & E \\ \hline & 1 & O & N & E & Y \end{array}$$

[1] Invented by Henry Dudeney, published in Strand Magazine in 1924.

Looking at the fourth column, we see that O can be either zero or one. In the latter case, one is carried over from the third column, so $E+1 \geq 9$. $E \neq 9$ because $S=9$, so $E=8$. If this is the case, then one is carried over from the second column and $N=0$. In the second column, this gives $0+R=10+E=18$, so $R=18$ and is a single digit number at the same time. Thus, $O=0$ and $E=N-1$.

		9	N-1	N	D
+		1	0	R	N-1
	1	0	N	N-1	Y

Suppose that one is not carried over from the first column to the second. Then $N+R=10+N-1$ implying $R=9$, an impossibility since $S=9$. Thus, one is carried over from the first column and $N+R+1=10+N-1$, giving us $R=8$ and $D+N-1 \geq 10$.

		9	N-1	N	D
+		1	0	8	N-1
	1	0	N	N-1	Y

Since $Y \neq 0,1$, the latter inequality can be strengthened to $D+N>12$. The single-digit number N can only take values $3,4,5,6,7$. Let us consider these possibilities case by case.

$N = 3$ implies $D > 9$.

$N = 4$ implies $D > 8$, but $D \neq 9$.

$N = 5$ implies $D > 7$, but $D \neq 8, 9$.

$N = 7$ implies $D > 5$, so $D = 6$ and $N - 1 = 6$.

Finally, $N = 6$ implies $D > 6$, so $D = 7$. $7 + 5 = 12$, so $Y = 2$. Here comes the solution.

$$
\begin{array}{rccccc}
 & & 9 & 5 & 6 & 7 \\
+ & & 1 & 0 & 8 & 5 \\
\hline
 & 1 & 0 & 6 & 5 & 2
\end{array}
$$

Problem 2 *Solve the following cryptarithm.*

$$
\begin{array}{r}
F\ O\ R\ T\ Y \\
+ \qquad T\ E\ N \\
T\ E\ N \\
\hline
S\ I\ X\ T\ Y
\end{array}
$$

Problem 3 *Solve the following cryptarithm.*

$$
\begin{array}{rcccccc}
 & M & O & T & H & E & R \\
+ & F & A & T & H & E & R \\
\hline
 & P & A & R & E & N & T
\end{array}
$$

Problem 4 *Solve the following cryptarithm.*

$$\begin{array}{r} O\ N\ E \\ +\ T\ H\ R\ E\ E \\ F\ O\ U\ R \\ \hline E\ I\ G\ H\ T \end{array}$$