Nimbers

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1 Impartial Games

Definition 1. A *combinatorial game* is a game that is

- two-player (two players alternate turns)
- complete-information (no face-down cards)
- deterministic (no randomness)
- well-founded (it's guaranteed to finish in a finite amount of turns)
- where the first player who cannot move loses, and thus the other player wins

We will start by studying *impartial* combinatorial games, which are games where the rules are the same for each player. It may still be better to be the player who goes first or second.

Problem 1. Here are some familiar examples of games. Do these meet the definitions, and are they impartial?

- Chess
- Checkers
- Poker
- Football (whatever that means to you)

When we study games, we will think about *positions* - a position is the state of the game, recording the possible moves for each player, right before a particular turn.

In an impartial combinatorial game, a position is called a winning position or first-player win if the first player is guaranteed to win if they use the right strategy, and a losing position or second-player win if the second player is guaranteed to win if they use the right strategy.

1.1 Nim

Perhaps the most classic impartial game to study is Nim. There are a few piles of objects (for our purposes, a few different groups of toothpicks on a Jamboard, but I recommend beans or coins if you play this IRL). On each turn, you take a positive number of toothpicks/beans/coins from *exactly one* pile. Play proceeds until someone empties the last pile, and thus wins, because there is no valid move for the other player.

*with thanks to John Conway and Alfonso Gracia-Saz

Problem 2. Use the toothpicks to play games of Nim. Start with 3 piles where the first pile has 5 toothpicks, the second pile has 6 toothpicks, and the third pile has 7 toothpicks. If your group has some people who have not played Nim before, they should try to play each other first, and then face more experienced players.

Problem 3. Say we're playing Nim with two piles - with m and n toothpicks respectively (m and n could be zero, in which case there are really 0 or 1 piles). For what values of m and n is this a winning or losing position?

Problem 4. Say you have an impartial game, and an algorithm for determining whether any given position is a winning or losing position. Describe a winning strategy for the game, assuming you get to choose whether you go first or second at the beginning.

Problem 5. Prove that each position in an impartial combinatorial game is guaranteed to either have the first or second player win - and is thus either a winning or a losing position. (Hint: If you have a game where every move has a winning strategy, show that the game itself has a winning strategy. Then if you have a game without a winning strategy, find a contradiction to well-foundedness.)

1.2 Other Subtraction Games

Subtraction games are variations on Nim - still impartial games. Let S be a set of positive integers. Then once again let us populate some piles with toothpicks, and once again you remove them from one pile at a time, but now you are only allowed to move a number of toothpicks in S.

Problem 6. (a) Let $S = \{1, 2\}$. Set up the game so that pile 1 has 5 toothpicks, pile 2 has 6 toothpicks, and pile 3 has 7 toothpicks.

(b) Let $S = \{1, 4\}$. Set up the game so that pile 1 has 5 toothpicks, pile 2 has 6 toothpicks, and pile 3 has 8 toothpicks.

(c) Let $S = \{1, 4\}$. Set up the game so the only pile has 5 toothpicks.

Feel free to mix and match S with the configuration, and generally play games as much as you want until you get a feel for the strategy.

1.3 Adding Games Together

I want to teach you a new board game. It's called *Chess + Checkers*. To play, we sit down on opposite sides of a table, and set up a chessboard and a checkerboard side-by-side. On each of our turns, we pick one of the boards and do a move in that board. As if chess and checkers aren't hard enough (well, maybe checkers isn't hard enough), now you have to decide whether to respond to my threat in chess, or get ahead in checkers.

In general, we can add any two combinatorial games G_1, G_2. In the new game, $G_1 + G_2$, each player gets to move in only one game at a time, and the person who runs out of moves in *both* games loses. (That is, if G_1 is finished first, no matter who won G_1, the winner of G_2 is the overall winner.)

Problem 7. Convince yourself that the sum of two impartial games is impartial, and the sum of two games of Nim is also a game of Nim (how many piles are there? of what sizes?).

Problem 8. Is this addition commutative? Is it associative? (Reminder: Commutativity means that $a+b=b+a$ for all a, b. Associativity means that for all a, b, c, we have $(a+b)+c=a+(b+c)$.)

Problem 9. Let G be an impartial game. Who wins $G+G$?

1.4 Sprague-Grundy Values

Definition 2. Let G be an impartial game with only finitely many valid moves at each position (like, for instance, Nim with finitely many toothpicks). Define the *Sprague-Grundy* value of G (written $\mathrm{SG}(G)$) recursively as follows:

- The empty game (with no valid moves) has a Sprague-Grundy value of 0.
- Calculate the Sprague-Grundy value of each position you can legally move to. Then let the Sprague-Grundy value be the smallest natural number that isn't on that (finite) list. We take the natural numbers to include 0 in this worksheet.

Problem 10. For the empty game, who has the winning strategy and why?

Problem 11. For a natural number n, let $*n$ represent a Nim game with one pile of n toothpicks (so $\mathrm{SG}(*0)=0$). What is the Sprague-Grundy value of $*n$?

Problem 12. Let's try a different Nim-ish game. There is one pile of n toothpicks, and at each turn, you must remove either 1 or 4 toothpicks. What is the Sprague-Grundy value of this game for each n? For which values of n is this a winning or losing position?

Problem 13. Prove that an impartial game is a second-player win if and only if its S-G value is 0.

This even gives us a winning strategy! For finite-move impartial games, we can recursively determine any position's Sprague-Grundy value, and then we know whether that position is a winning or losing position! Then it's just a matter of trying to move to losing positions whenever possible.

1.5 Nim Addition

To calculate the Sprague-Grundy values of Nim games, and thus to develop a winning strategy for Nim, we will define a special kind of addition on the natural numbers, called *Nim-addition*, and written with $\oplus$. We define it so that $m \oplus n = \mathrm{SG}(*m + *n)$, that is, to add m and n this way, we find the Sprague-Grundy value of a Nim game with one pile of height m and one pile of height n. It turns out that for any impartial games G, H, $\mathrm{SG}(G+H) = \mathrm{SG}(G) \oplus \mathrm{SG}(H)$.

Problem 14. Calculate a Nim-addition table for the numbers $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

Problem 15. Form a guess for calculating $m \oplus n$ in general.

Hint: Use your table for inspiration, and try writing m and n in binary.

Problem 16. Go back to your earlier Nim games, and see if you can use Nim-addition to win!

Problem 17. Prove that your formula for $m \oplus n$ works.

Hint: Use a kind of induction. Assume that this formula works for calculating $a \oplus b$ whenever $a + b < 2^m + 2^n$. Then check that $2^m \oplus 2^n$ follows your formula.

Problem 18. For what values of n is the set $\{0, 1, 2, \ldots, (n-1)\}$ closed under Nim-addition? (That is, if $i, j < n$, then $i \oplus j = k$ for some $k < n$.)

1.6 Bonus

Problem 19. Let's look at a modification of the earlier subtraction problem. There is one pile of n toothpicks, and at each turn, you must remove a number of toothpicks which is a power of 4. What is the Sprague-Grundy value of this game for each n? For which values of n is this a winning or losing position?

2 Multiplying Nimbers

We'll now define another game that will allow us to extend our arithmetic on nimbers.[1] We play on a grid, where we place counters on lattice points of the grid. As before, this will be an impartial game, where the players take turns playing legal moves until someone makes the last possible legal move and thus wins. To make a move, pick a counter and do one of the following:

- Remove it.
- Move it to the left (as many spaces as you want).
- Move it down (as many spaces as you want).
- If this counter is at the upper-right corner of some rectangle, remove this counter and add counters to the other three corners. We will call this the rectangle rule.

If after this, you end up with two counters on the same grid point, remove both of them.

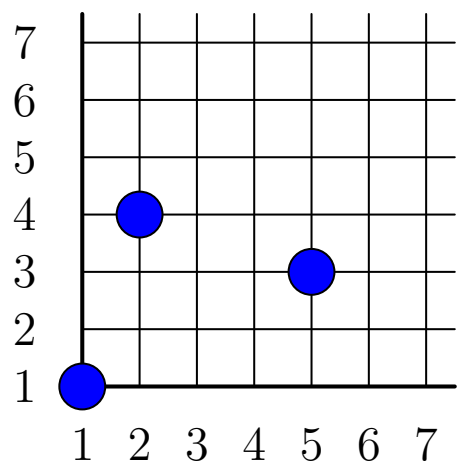
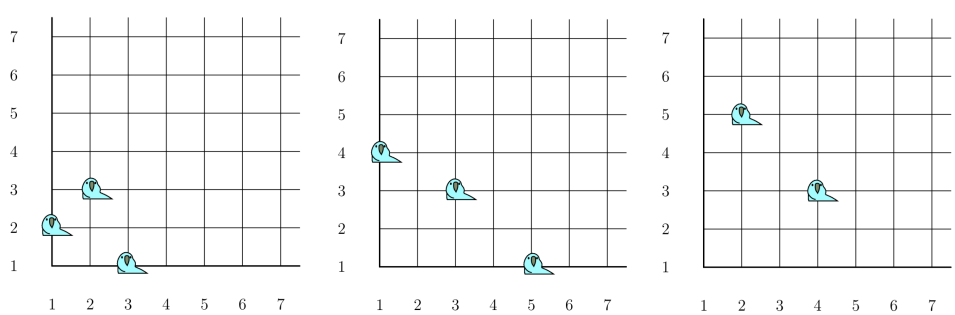


Figure 1: A valid setup with 3 counters

Problem 20. Try playing the game with the following setups. We will find the strategy soon, but think about the following questions: Who will win each game, and is it possible to change the winner by adding a single counter on the diagonal?



Problem 21. Let's say we have a configuration of this game where all the counters are in the leftmost column (or equivalently the bottom row). Explain how to calculate its Sprague-Grundy value. What does this have to do with Nim?

[1] This game comes from this blog, and is based on a game by Lenstra.

Problem 22. Explain how if you added another row and column (each labelled 0), you could summarize all moves as just the rectangle rule, with extra step that at the end of each move, you remove all counters on row 0 or column 0.

Definition 3. If $m, n \in \mathbb{N}$, we can define multiplication on nimbers: $m \otimes n$ is the S-G value of the game with moves $\{*\big([m \otimes n'] \oplus [m' \otimes n] \oplus [m' \otimes n']\big) : m' < m, n' < n\}$. We take $0 \otimes n = 0$ and $m \otimes 0 = 0$.

This multiplication is associative, $k \otimes (m \otimes n) = (k \otimes m) \otimes n$, and distributes with Nim-addition, $k \otimes (m \oplus n) = k \otimes m \oplus k \otimes n$, but we will not prove this.

Problem 23. Check that $1 \otimes n = n$ for all n and $m \otimes 1 = m$ for all m.

Problem 24. If $m, n \in \mathbb{N}$, place a counter at the position (m, n) on the grid in our new game. Show that the S-G value of this game is $m \otimes n$.

Problem 25. Calculate a Nim-multiplication table for the nimbers $\{0, 1, 2, 3\}$.

Problem 26. Prove that for all $a, b > 0$, $a \otimes b \neq 0$. Which player has a winning strategy for $*(a \otimes b)$?

Problem 27. It turns out that for all $m \neq n$, $2^{2^m} \otimes 2^{2^n} = 2^{2^m + 2^n}$, and $2^{2^n} \otimes 2^{2^n} = 3 \otimes 2^{2^n - 1}$.

- Use these facts (and distributivity and such) to calculate $8 \otimes 8$.
- Explain how you can use these rules to calculate Nim-multiplication in general.
- Show that the set $\{0, 1, \ldots, 2^{2^N} - 1\}$ is closed under Nim-multiplication.

Problem 28. Assuming that $\{0, 1, 2, \ldots, (2^{2^N} - 1)\}$ is closed under Nim-multiplication, prove that for all $0 < n < 2^{2^N}$, there exists $m < 2^{2^N}$ such that $m \otimes n = 1$.

Problem 29. Prove that any arrangement in this game can be turned into a second-player win by adding or removing a counter from the diagonal. (Hint: What does this mean in algebraic terms? Show also that every position with only a single counter on the diagonal has a different value.)

Problem 30. Prove the assertions we've made so far by induction. Specifically, show the following by induction on N:

- If $m < n < N$, then $*2^{2^m} \otimes *2^{2^n} = *2^{2^m + 2^n}$.
- If $n < N$, then $*2^{2^n} \otimes *2^{2^n} = *(3 \otimes 2^{2^n - 1})$.
- The set $\{*0, *1, \ldots, *(2^{2^N} - 1)\}$ is closed under multiplication.