Combinatorics and Permutations

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1 Combinatorics

Combinatorics is a branch of math focused around counting! Counting is a powerful tool that allows us to compute probabilities, existence of certain mathematical objects, how many options for passwords under certain criterion, and much more! Let's start with a few definitions and examples.

Definition 1 (Permutation)**.** A permutation is an ordered rearrangement of elements.

Example 2. The set of permutations of the word dog:

$$\{DOG, ODG, GOD, DGO, OGD, GDO\}$$

Notice that this set has 6 elements. Is there anything special about the number 6?

Problem 1. Calculate how many permutations of the following words exist:

MATH

TREES

MISSISSIPPI

What is the problem with repeated letters?

Problem 2. Suppose a class has 100 students. The class board is electing 3 students at random to be part of the executive board. There are three distinct positions: president, vice president, and advisor. How many different ways can they choose an executive board? How many different ways can they choose if there were no distinct positions on the board?

Problem 3. Suppose I have a bag of 20 distinct numbers, and I am going to draw 6 numbers. How many different hands of numbers can I draw?

A very useful tool in combinatorics is the **choose** operator. If you want to pick k objects out of a set of n objects, this would be *choosing* k objects from n objects.

Example 3. If Kevin wants to pick out 3 distinct prime numbers that are less than 20, we can write out the set of possible options:

$$\{2, 3, 5, 7, 11, 13, 17, 19\}$$

Let's say that Kevin will permute this set and then pick the first 3 elements of the permuted set. Then, the order of these numbers does not matter since they will be the same 3 numbers. Similarly, the order of the rest of the numbers in the list does not matter. So the number of ways to permute, say $\{2, 3, 5\}$ is $3! = 3 * 2 * 1 = 6$, and the number of ways to permute $\{7, 11, 13, 17, 19\}$ is $5! = 5 * 4 * 3 * 2 * 1 = 720$. So these are the number of ways we can reorder these sets to have the same elements in them. So in total, since we can permute the entire set in 8! ways, then dividing by the number of permutations that give the same result, we have

$$\frac{8!}{3! * 5!} = 56$$

Mathematically, this is represented by

$$\binom{8}{3} = \frac{8!}{3! * 5!}$$

In general, we have the formula

$$\binom{n}{k} = \frac{n!}{k! * (n-k)!}$$

Problem 4. Prove that $\binom{n}{k} = \binom{n}{n-k}$. Then, try to state in words why this identity makes sense.

Many proofs in combinatorics can be done mathematically. But writing down statements that are equal does not help you understand the whole picture. We can think of some problems **combinatorially** by explaining with an example why the statement should be true. When you see $\binom{n}{k}$ try to think of a situation where you have to choose k objects from n objects. In the following problems, try give an algebraic and combinatorial proof of the statement. It often helps to think about the situation in different cases.

Problem 5. Prove that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

(Hint: Imagine you have n objects, and you want to choose k of them. Then, imagine you add in a new object to the original n objects. Where are the possible places to put the new object while still choosing k total objects?)

Algebraic:

Combinatorial:

Problem 6. Prove that $\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + ... + \binom{n}{k}$

(Hint: Imagine you are choosing k+1 numbers from the set $[n+1] = \{1, 2, ..., n+1\}$. What options do you have for the smallest number that you choose?)

Algebraic:

Combinatorial:

Problem 7. Show that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = 2^n$

(Note: If this takes too long, skip this problem and return to it after the page section of the worksheet)

Algebraic:

Combinatorial:

Definition 4 (Set)**.** A **set** is a collection of **elements** denoted by curly brackets $\{\_\}$. An element can be any sort of object. The **cardinality** or **size** of a set, X is the number of elements, demoted by $|X|$. A set Y is a **subset** of X if every element of Y is an element of X. We denote x is an element of set X by $x \in X$, and Y is a subset of X by $Y \subset X$.

Example 5. Some examples of sets are:

1. {Kevin, Yan, Chris, Matthew}

2. $\{\} = \emptyset$ (the set with no elements)

3. $\{\emptyset\}$ (How is this different from 2.?)

4. The set of Natural Numbers:

$$\mathbb{N} = \{1, 2, 3, ...\} \text{ (No, I do not include zero)}$$

5. The set of Integers:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

6. The set of the first n Natural Numbers:

$$[n] = \{1, 2, ..., n\}$$

We will be focusing on this set throughout the worksheet.

To translate the previous problem into a combinatorics problem, consider the set $[n]$. The left hand side asks us to find the number of ways we can choose $0, 1, ..., n$ elements from the set $[n]$. We can think about this in terms of subsets. We want to find the number of subsets of size $0, 1, ..., n$. Since $|[n]| = n$, then these are all possible sizes of subsets. So we want to find how many subsets of $[n]$ there are. Now, we draw n spaces, one for each of the elements in $[n]$.

1	2	3	4	5	6	...	n

Pick any subset of $[n]$. We will use the set $\{1, 2, 3, 5, n\}$. In each of the spaces, we will write 1 above any element that is in the subset and 0 above every element that is NOT in the subset.

1	1	1	0	1	0	...	1
1	2	3	4	5	6	...	n

Notice that this generates a binary string for each subset of $[n]$.

Problem 8. How many binary strings of length n exist?

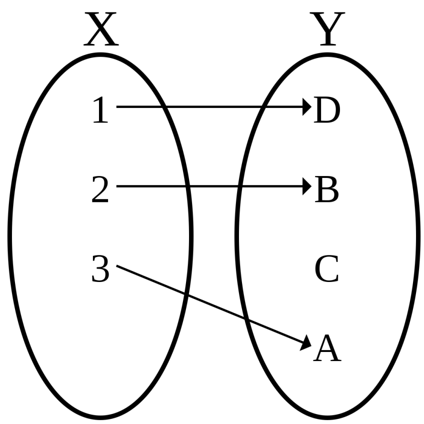
Problem 9. Let X and Y be two subsets of $[n]$ with $X \neq Y$. Prove that the binary strings, X_2 and Y_2 respectively, generated by the above construction are distinct ($X_2 \neq Y_2$). Conclude that the number of subsets of $[n]$ is 2^n.

2 Functions

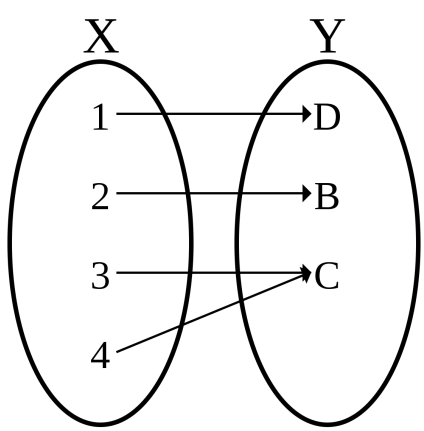
Let's introduce some properties of functions that will be useful for the rest of the worksheet.

Definition 6 (Domain, Codomain)**.** A function f is defined as a map from a **domain** to a **codomain**. We write $f : X \to Y$ where X is the domain and Y is the codomain. Essentially, f takes elements in set X and maps them to elements in set Y. The **range** of f is all of the points y in Y where for some x in X, $f(x) = y$. We write $f(X)$ to represent the **range** of the function.

Definition 7 (Injective)**.** A function $f : X \to Y$ is said to be **injective** if for x_1, x_2 in X, we have $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. We also call this **one-to-one**.



Definition 8 (Surjective)**.** A function $f : X \to Y$ is said to be **surjective** if the codomain equals its range, $Y = f(X)$. Or, for each y in Y there exists an x in X such that $f(x) = y$. We also call this **onto**.



Definition 9. A function $f : X \to Y$ is said to be **bijective** if it is both **injective** and **surjective**.

Problem 10. Label each of the following functions as **injective**, **surjective**, **bijective**, or **none of these**.

1. $f : \{1, 2, 3\} \to \{4, 5, 6\},\ f(x) = x + 3$

2. $f : \mathbb{N} \to \mathbb{N},\ f(x) = x + 1$

3. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$

4. $f : [0, \infty) \to \mathbb{R},\ f(x) = x^2$

5. $f : \mathbb{R} \to [0, \infty),\ f(x) = x^2$

6. $f : [0, 1] \to [0, 1],\ f(x) = x^2$

Notice how the domain and codomain change properties of the function itself. Therefore, we typically consider the domain and codomain part of the defining features of a function.

Problem 11. Let n, m be in $\mathbb{N}$ and X, Y be two sets with $|X| = n$ and $|Y| = m$. Let $f : X \to Y$. If $n > m$, can f be injective? Explain. If $n < m$, can f be surjective? Explain.

Problem 12. If $f : X \to Y$ is bijective, how would we define a function that goes backwards from Y to X? In other words, how would we define an inverse function? Then, if f is injective but NOT surjective, how would we construct an inverse function?

(Hint: Remember, we have to consider a domain and codomain that makes sense when we construct our function!)

Let's study functions of the form $f : [n] \to [n]$.

Problem 13. How many injective functions $f : [n] \to [n]$ exist? How many such surjective functions? How many such bijective functions exist?

What is the relationship between the number of each type of function in the above problem? Can you explain why?

Problem 14. Let $f : [n] \to [n]$. Show that f is injective if and only if f is surjective.

Problem 15. Use the above problems to conclude that there is a bijection between the permutations of $[n]$ and bijections in the form $f : [n] \to [n]$.

Example 10. Let $f : [8] \to [8]$ be defined as follows:

$$f(1) = 4, f(2) = 2, f(3) = 8, f(4) = 5, f(5) = 6, f(6) = 1, f(7) = 3, f(8) = 7$$

Then each element in [8] maps to a unique element in [8] so this f is bijective. Now, let's repeatedly apply f to each element. For example:

$$f(1) = 4 \to f(f(1)) = f(4) = 5 \to f(f(f(1))) = f(f(4)) = f(5) = 6$$
$$\to f(f(f(f(1)))) = f(f(f(4))) = f(f(5)) = f(6) = 1$$

What would happen if we continued this process?

Problem 16. Repeatedly apply f to each of the remaining elements similar to the above example, ending when you return to the original element.

1. $f(2)$

2. $f(3)$

3. $f(4)$

4. $f(5)$

5. $f(6)$

6. $f(7)$

7. $f(8)$

Notice that we have these cycles:

$$1 \to 4 \to 5 \to 6 \to 1$$

$$2 \to 2$$

$$3 \to 8 \to 7 \to 3$$

Let's assign a notation to these cycles.

$$1 \to 4 \to 5 \to 6 \to 1 = (1456) \text{ this is a 4-cycle}$$

$$2 \to 2 = (2) \text{ this is a 1-cycle}$$

$$3 \to 8 \to 7 \to 3 = (387) \text{ this is a 3-cycle}$$

Then we will denote our function f by the product of cycles:

$$f(x) = (1456)(2)(387)$$

For convenience, if an element gets mapped to itself, for instance 2 in the above example, we may omit it from the product since it will only be part of its own 1-cycle. So we can write:

$$f(x) = (1456)(387)$$

.

If the function is the product of ONLY 1-cycles, for instance

$$g(x) = (1)(2)(3)(4)(5)(6)(7)(8)$$

then this maps every element back to itself. We call this the **identity** function denoted

$$g(x) = x \iff g(x) = id$$