The Disjunctive Normal Form and Geometry of Cubes

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Adapted from a series of worksheets by Oleg Gleizer

1 The Disjunctive Normal Form

Recall *de Morgan's Laws* from last week:

$$\neg(A+B) = \neg A \times \neg B \text{ and } \neg(A \times B) = \neg A + \neg B$$

For every Boolean algebra expression, we can use de Morgan's Laws to write it in an equivalent form that contains no negations of composite statements. Such an expression can be represented as a sum of products of Boolean variables and their negations. It is called the *disjunctive normal form*, or the DNF, of the original expression. The name originates from the word *disjunction*, another way to call the logical addition. By the way, another name for logical multiplication is *conjunction*.

The following is an example of a disjunctive normal form. Can you see why the two sides of this equation are equal?

$$\text{DNF}(\neg(AB + C\neg D) + \neg B + \neg C) = \neg AD + \neg B + \neg C$$

Problem 1 *Can a simplified expression possibly have a product containing both a Boolean variable and its negation? Why or why not?*

Problem 2 *Find the DNF of the following two expressions. Simplify if possible.*

- $XY + \neg(YZ + \neg ZX) =$

- $A\neg BC + \neg(\neg ABC + \neg AB + C) =$

A DNF formula is in the *full disjunctive normal form*, or FDNF, if each of the variables appears exactly once in every product. For example, let us consider the expression $AB + BC$. It is in the DNF, but not in the FDNF. There is no variable C, or its negation, in the first product. There is no variable A, or its negation, in the second. Let us use the identities $A + \neg A = 1$ and $C + \neg C = 1$ to bring the expression to the FDNF.

$$AB + BC = AB(C + \neg C) + BC(A + \neg A) = ABC + AB\neg C + \neg ABC$$

Problem 3 *Find the FDNF of the following four Boolean algebra expressions.*

- $A + B =$

- $AB + \neg C =$

- $A + \neg B + C =$

- $\neg(AB + C\neg D) + \neg B + \neg C =$

2 FDNF and Geometry of Cubes

Let us use a 2-dimensional (2D) cube, a.k.a. a square, to visualize Boolean algebra expressions with two simple statements. As an example, let us take the first expression from Problem 3. The first step is to find the FDNF of the expression.

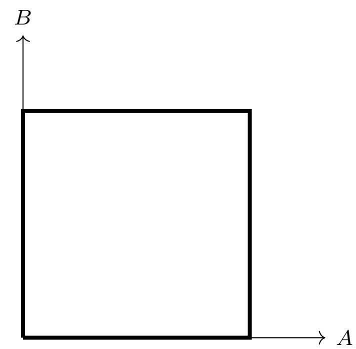
$$\text{FDNF}(A + B) = AB + A\neg B + \neg AB$$

The second step is to draw a 2D cube that is placed in a 2D coordinate system having the following features.

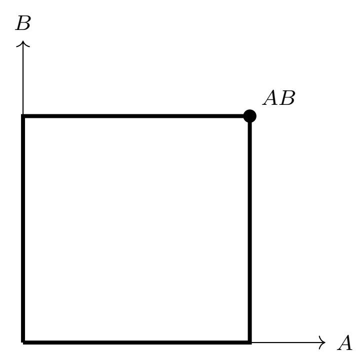
1. The origin coincides with one of the cube's vertices.

2. The edges of the cube having the origin as a vertex span the coordinate axes.

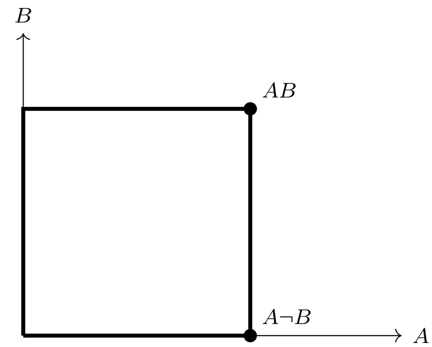
3. The axes are named by the simple statements.



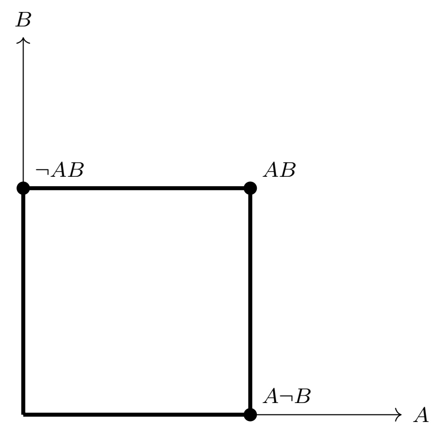
Let us take each product from the FDNF expression above and consider it as a word that tells how to move through the vertices of our 2D cube starting from the origin. The word AB is understood as the instruction to move in the direction of A, then to move in the direction of B. This brings us to the vertex opposite to the origin.



The next summand, the word $A\neg B$, is the instruction to move in the direction of A and not to move in the direction of B.



Finally, the last summand, the word $\neg AB$, tells us not to move in the direction of A, but to move in the direction of B.

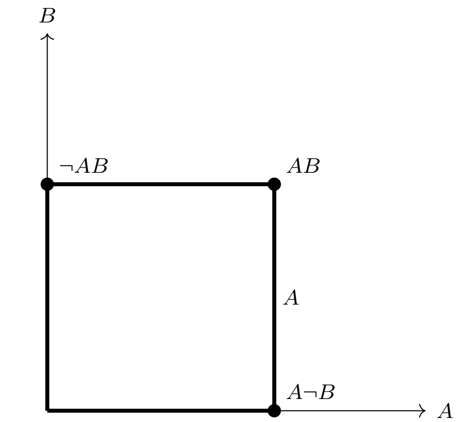


This way, the FDNF formula

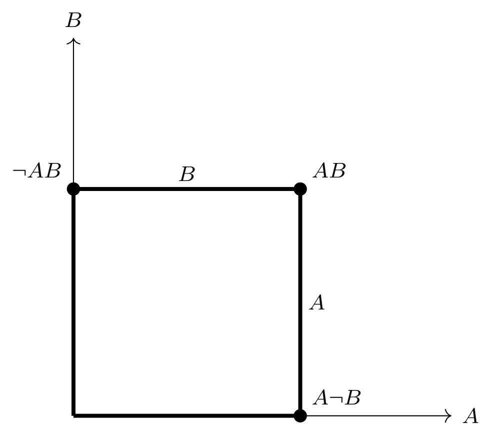
$$AB + A\neg B + \neg AB \tag{1}$$

gets represented by the three vertices at the above picture. Let us see how the picture helps to simplify 1 to the original $A + B$.

An edge of a cube can be described by a pair of its vertices. For example, the edge of our 2D cube having the vertices AB and $A\neg B$ is the edge parallel to the vertical axis, but not lying on it. We will consider an edge connecting two vertices that are marked with products of Boolean variables, or their negations, as a graphical representation of the sum of the corresponding products. For example, the edge connecting the vertices AB and $A\neg B$ represents the sum $AB + A\neg B$. Since $AB + A\neg B = A(B + \neg B) = A$, we mark the corresponding edge with A.



Similarly, the edge having the vertices $\neg AB$ and AB corresponds to the logical sum $\neg AB + AB = B(\neg A + A) = B$.



This way, the entire simplification

$$\neg AB + AB + A\neg B = A + B$$

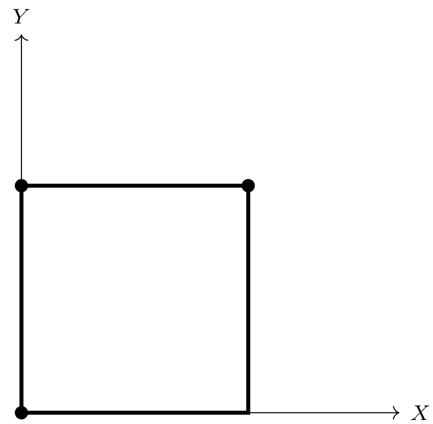
appears right in front of our eyes!

Question 1 *The product AB was used to simplify the sum $AB + A\neg B$. Why do you use it again in the $AB + \neg AB$ sum simplification?*

Answer We can use multiple copies of the word AB due to the fact that in Boolean algebra, $AB + AB + \ldots = AB$.

Problem 4

- *Write down the FDNF expression that corresponds to the following marked vertices of a 2D cube.*

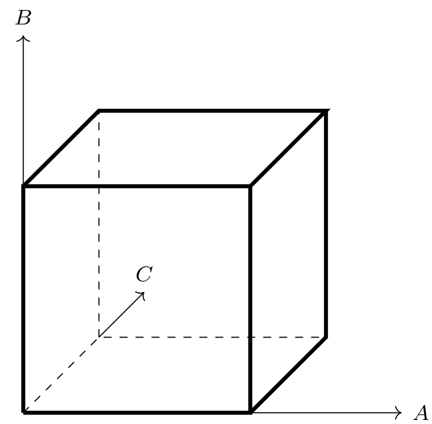


- *Use the edges of the cube to simplify the expression.*

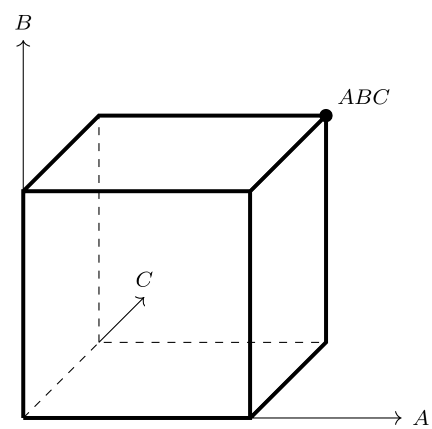
- *Simplify the expression algebraically and see if you get the same result.*

Let us use the geometric approach to simplify the following FDNF expression.

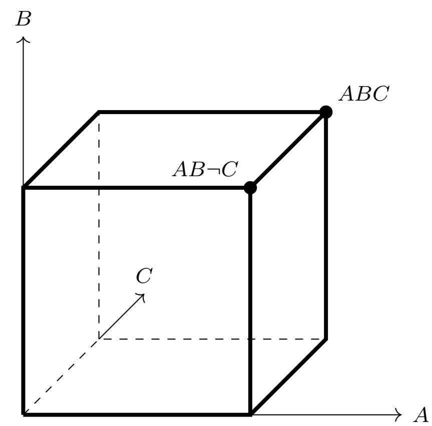
$$ABC + AB\neg C + A\neg BC + \neg ABC + \neg A\neg BC \tag{2}$$



The first word, ABC, instructs us to move along A, B, and C, bringing us to the vertex opposite to the origin.



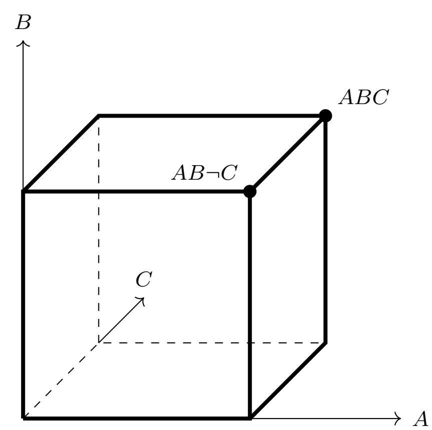
The second word, $AB\neg C$, tells us to move along A and B, but not to move along C.



Problem 5 *Mark the rest of the products from formula 2, copied for your convenience here,*

$$ABC + AB\neg C + A\neg BC + \neg ABC + \neg A\neg BC$$

on the picture below.

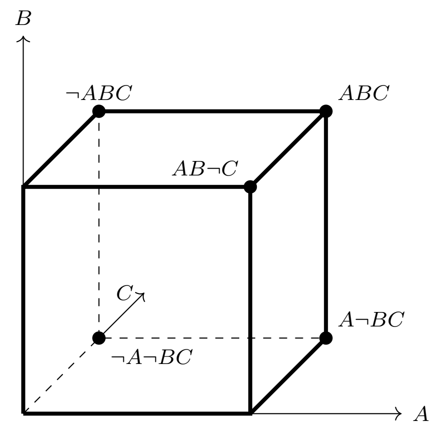


Then try to simplify the formula using the edges of the cube. The solution is on page 10. Don't look there yet!

All the products from formula 2, copied for your convenience one more time,

$$ABC + AB\neg C + A\neg BC + \neg ABC + \neg A\neg BC$$

are now marked at the picture below.

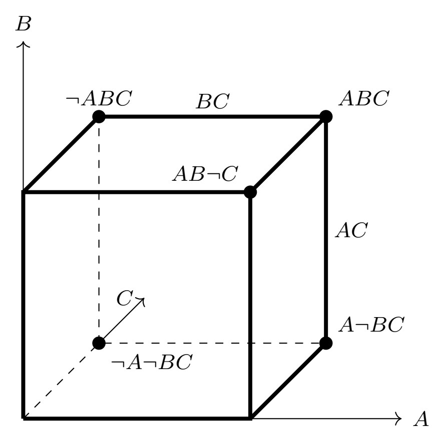


Let us simplify "along the edges". The edge having the vertices ABC and $A\neg BC$ corresponds to the sum

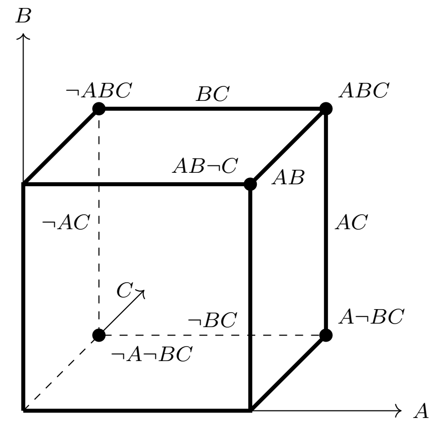
$$ABC + A\neg BC = A(B + \neg B)C = AC.$$

The edge having the vertices ABC and $\neg ABC$ corresponds to the sum

$$ABC + \neg ABC = (A + \neg A)BC = BC.$$



Problem 6 *Mark the remaining edges of the 3D cube above with the simplified sums of the products corresponding to the vertices. The solution is on the next page. Don't look there before you try it yourself!*



The sum 2 simplifies to the following.

$$AC + BC + \neg AC + \neg BC + AB \tag{3}$$

Note that the first four products correspond to all the edges of a 2D face of our 3D cube. Let us first add up these four.

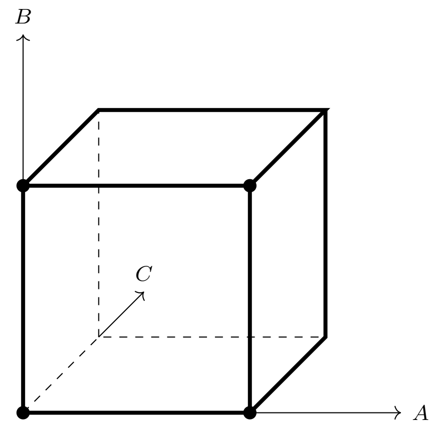
$$AC + BC + \neg AC + \neg BC = (A + \neg A + B + \neg B)C = C$$

The face in consideration corresponds to the Boolean expression C just like the edges corresponded to AC, BC, etc. Finally, formula 3, as well as the original 2, simplify to

$$AB + C$$

which cannot be simplified any further.

Problem 7 • *Write down the FDNF expression that corresponds to the following marked vertices of a 3D cube.*

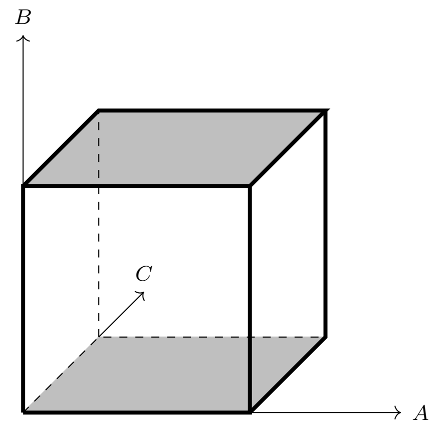


• *Simplify the expression as much as you can.*

• *What Boolean algebra expression marks the front face of a 3D cube?*

• *This face is opposite to the face C considered in the example above. How can we see it from the Boolean algebra expressions corresponding to the faces?*

Problem 8 *Guess the Boolean algebra expressions that mark the two shaded opposite faces of a 3D cube. Hint: what coordinate axis are they perpendicular to?*



Write down the FDNF expressions that correspond to the sums of the vertices of each of the faces, simplify the sums and check your guess.

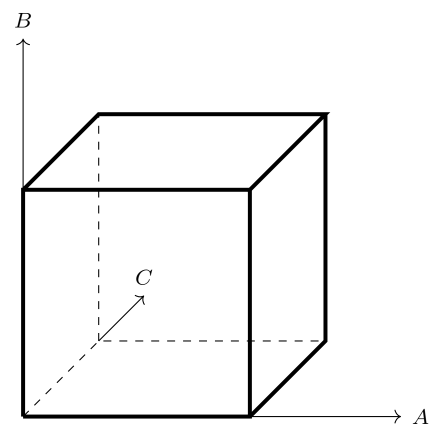
- *The top face:*

- *The bottom face:*

Problem 9 *Bring the following expression*

$$ABC + A\neg BC + A\neg B\neg C + B\neg C$$

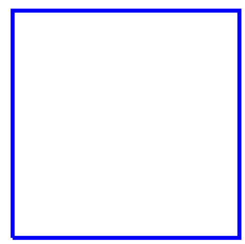
to the FDNF form. Then use the geometric approach to simplify.



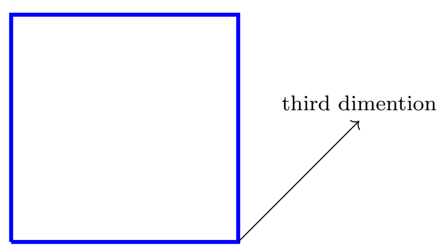
To deal in a similar way with an expression that involves four simple Boolean algebra statements, we first need to learn how to draw a 4D cube and to study some of its properties.

To understand how to draw a 4D cube, let us closely examine the following two ways of drawing a 3D cube on a 2D sheet of paper and then generalize to the fourth dimension.

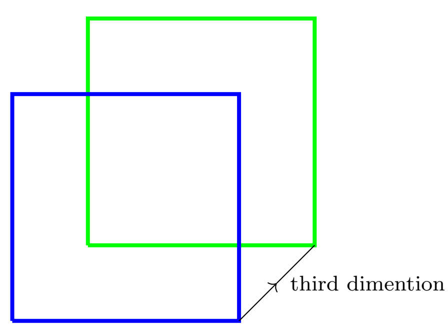
Here is the first way. Let us draw the front face of a would-be 3D cube. This is a 2D cube a.k.a. a square.



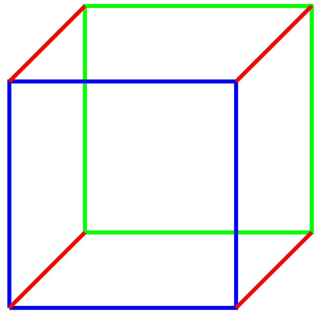
Since the sheet of paper we use for drawing is 2D, we don't really have the third dimension we need. However, we can pretend that the arrow below points in the third dimension, can't we?



Dragging the front face of a would-be 3D cube in the "third dimension" gives us the rear face of the 3D cube.

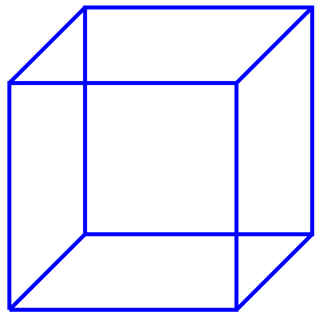


The last thing to do is to connect the corresponding vertices.

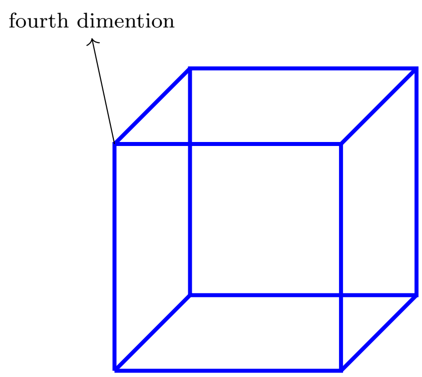


Problem 10 *Draw a 4D cube in the space below. You will find our drawing, explained step-by-step, on the following pages. Don't look there yet!*

Let us first draw the front face of a would-be 4D cube, a 3D cube.



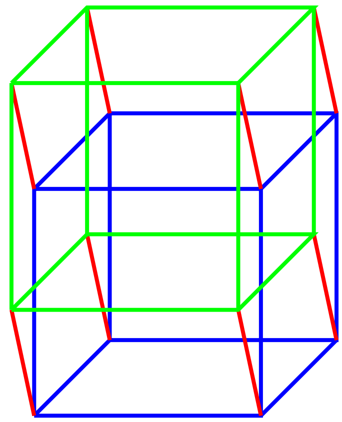
Let us further pretend that the arrow below points in the fourth dimension.



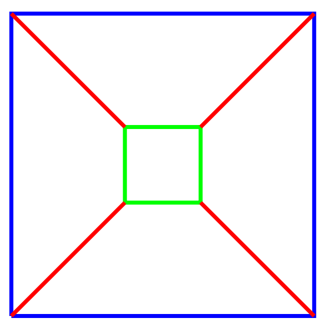
Question 2 *Where is this fourth dimension?*

Answer It doesn't matter! We can draw the third dimension on a 2D sheet without ever knowing where the third dimension is, can't we? The same trick works for the fourth dimension (as well as for the fifth, sixth, and so on).

Let us take the front face of our 4D cube, the 3D cube we have drawn above, and drag it in the "fourth" dimension. Here comes a 4D cube (a.k.a. a *hypercube* or *tesseract*), or rather its 2D picture.

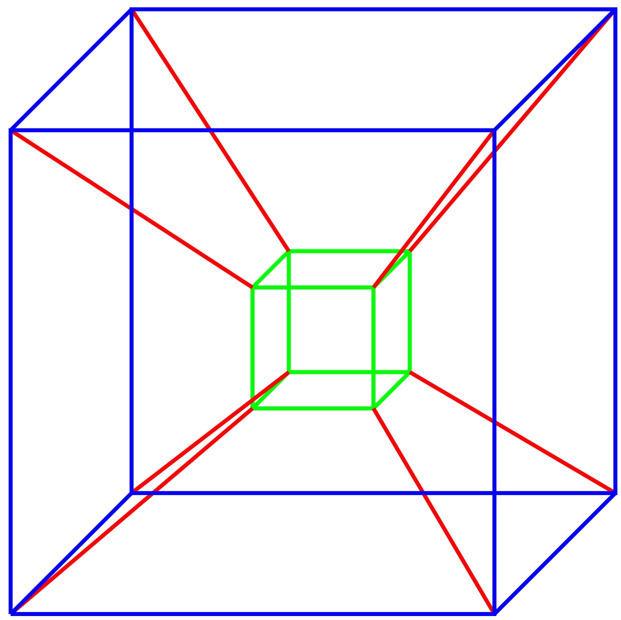


A different way to draw a 3D cube, as well as a 4D one, is to use *perspective*. The main feature of perspective is that an object further away from an observer appears smaller than the same object next to the observer. For example, an airplane up in the sky seems to be way smaller than the same plane on a runway. Perspective makes the rear face of the 3D cube below look smaller than the front face, although in reality they have the same size. What's more, the trapezoids joining the squares are all squares as well, equal in size to the front and rear faces of the 3D cube. The perspective just makes them look different!



Problem 11 *Use perspective to draw a 4D cube in the space below. Our drawing is on the next page. Don't look there yet!*

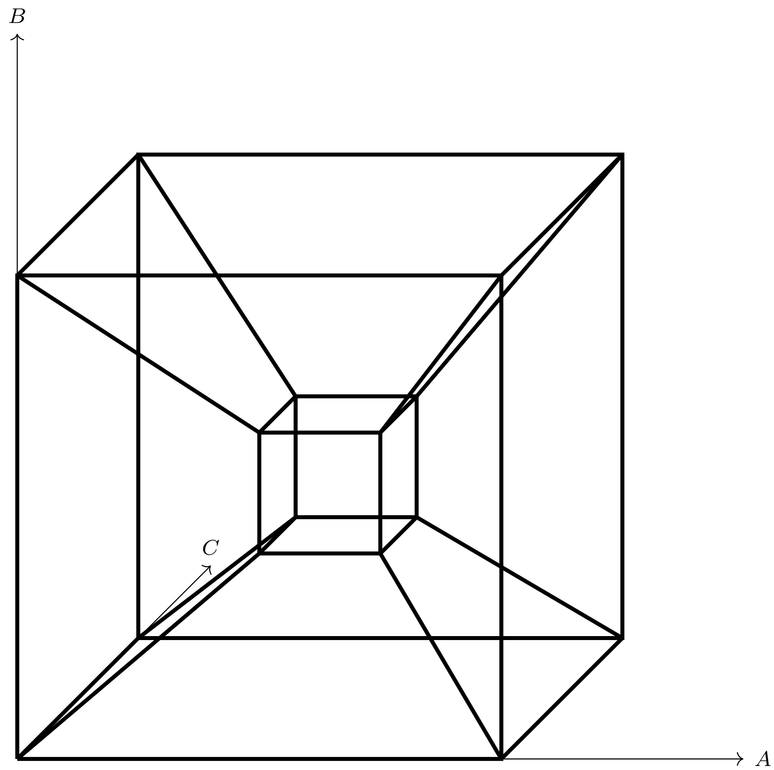
On the picture below, the smaller 3D cube stick out not inside the larger one, but outside of it in the fourth dimension. The 3D cubes are of the same size, but perspective makes them look different. The six 3D truncated pyramids joining the 3D cubes are also 3D cubes of the same size, deformed by perspective.



Problem 12 *Count the number of vertices, edges, 2D and 3D faces of a tesseract and fill out the following table.*

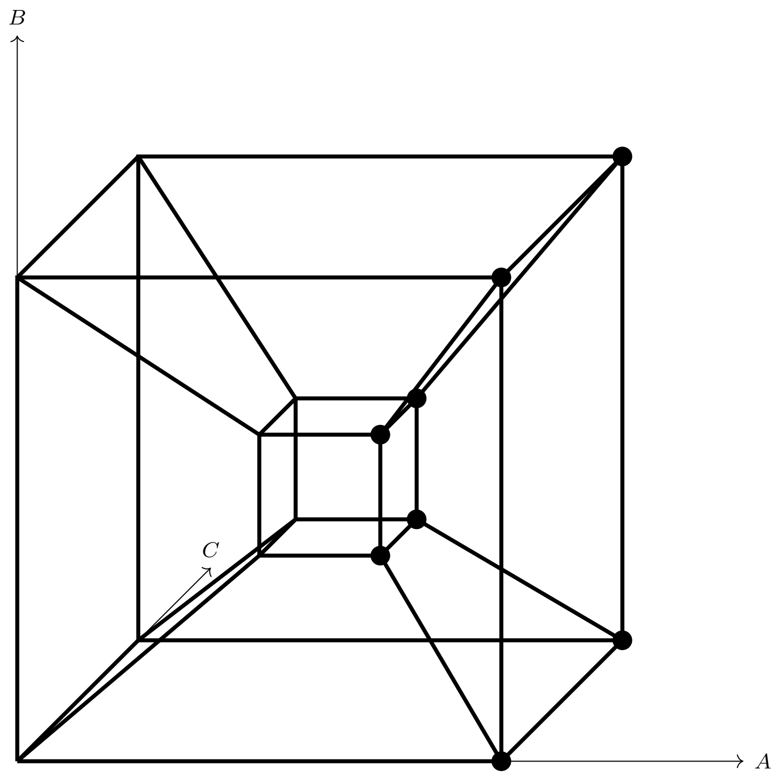
vertices	*edges*	*2D faces*	*3D faces*

Problem 13 *Mark the vertex corresponding to the product $AB\neg CD$ on the picture below. Hint: the only way to get to the "smaller" 3D cube is to move in the direction of the fourth coordinate axis, D.*



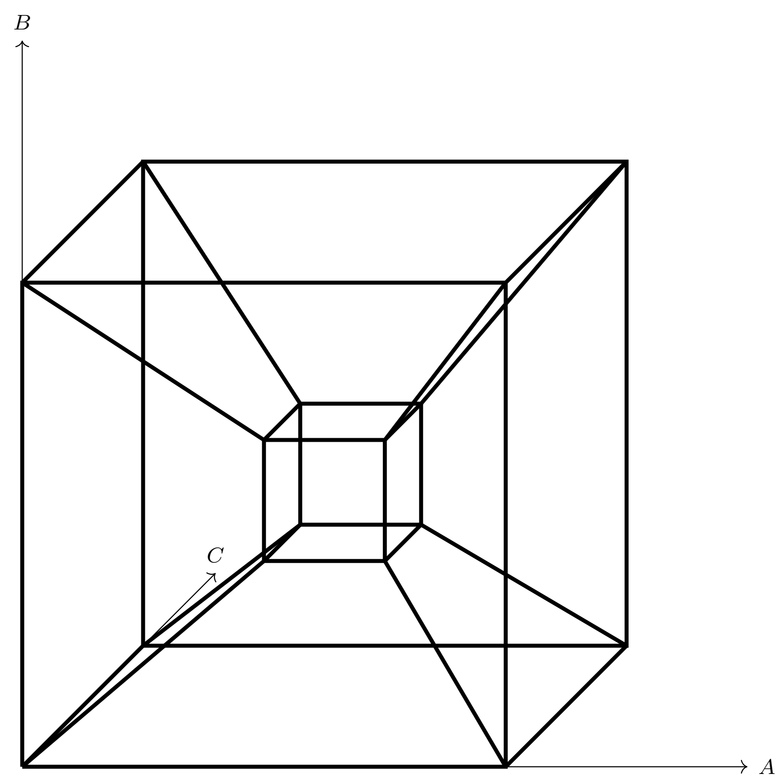
Problem 14 *Write down the FDNF sum corresponding to all the vertices of the "smaller" 3D cube and simplify it as much as you can. Guess what the same procedure would produce for the vertices of the larger 3D cube.*

Problem 15 *Write down the FDNF sum corresponding to the vertices of the hypercube marked below. Simplify the sum as much as you can. Guess what the same procedure would yield for the vertices of the opposite 3D face.*

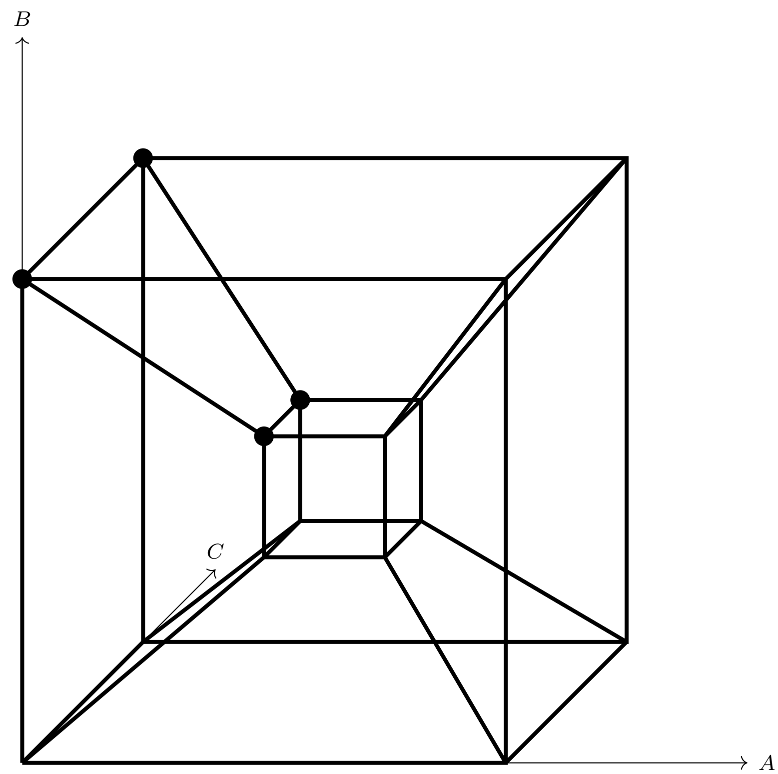


Problem 16 *Use the hypercube below to simplify the following Boolean algebra expression.*

$A\neg B\neg C\neg D + AB\neg C\neg D + A\neg BC\neg D + ABC\neg D + A\neg B\neg CD + A\neg BCD + AB\neg CD + ABCD + \neg AB\neg CD + \neg ABCD =$

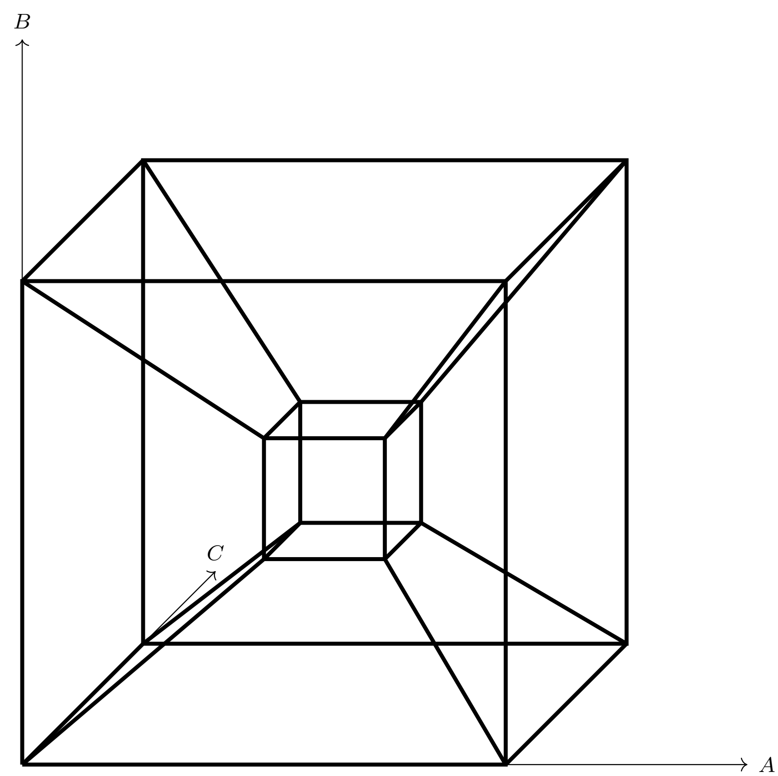


Problem 17 *Write down the FDNF sum corresponding to the vertices of the hypercube marked below. Simplify the sum as much as you can.*



Problem 18 *Use the hypercube below to simplify the following Boolean algebra expression.*

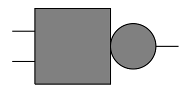
$\neg AB\neg C\neg D + \neg ABC\neg D + \neg AB\neg CD + \neg ABCD + AB\neg CD + ABCD =$



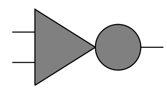
3 Bonus Section I: Universal Gates

Recall that the boolean operations we've been working with can be realized in hardware by the OR, AND, and NOT logic gates. The following logic gates are called *universal* because all other logic gates can be built using just one of the universal gates.

Nand gate



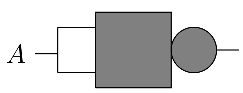
Nor gate

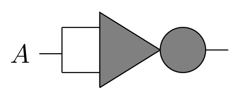


A	B	**Nand**(A, B)
0	0	1
1	0	1
0	1	1
1	1	0

A	B	**Nor**(A, B)
0	0	1
1	0	0
0	1	0
1	1	0

For example, the following circuits function as the **Not** gate.





Problem 19 *Please check the correctness of the statement above.*

Problem 20 *Realize the* **And** *gate as a circuit using the* **Nand** *gate only.*

Problem 21 *Realize the* **And** *gate as a circuit using the* **Nor** *gate only.*

Problem 22 *Realize the* **Or** *gate as a circuit using the* **Nand** *gate only.*

Problem 23 *Realize the* **Or** *gate as a circuit using the* **Nor** *gate only.*

4 Bonus Section II: Comparing to the Algebra of Polynomials

Problem 24 *If possible, please expand and then simplify the expressions below according to the rules of the corresponding algebras.*

Algebra of Polynomials	**Boolean Algebra**
$1 + 1 =$	$1 + 1 =$
$a + (-a) =$	$a + (\neg a) =$
$w \times (-w) =$	$w \times (\neg w) =$
$x(x + 1) =$	$x(x + 1) =$
$(x + y)(x + y) =$	$(x + y)(x + y) =$
$(x + (-y))(x + (-y)) =$	$(x + (\neg y))(x + (\neg y)) =$
$-(\underbrace{s + \ldots + s}_{\text{56 times}}) =$	$\neg(\underbrace{s + \ldots + s}_{\text{56 times}}) =$

If you finished everything above, there's another table to fill on on the next page!

Algebra of Polynomials	**Boolean Algebra**
$-(\underbrace{p \times \ldots \times p}_{\text{65 times}}) =$	$\neg(\underbrace{p \times \ldots \times p}_{\text{65 times}}) =$
$a \times (a + b) =$	$a \times (a + b) =$
$(x + y)(x + z) =$	$(x + y)(x + z) =$
$(x + y)(x + (-y)) =$	$(x + y)(x + (\neg y)) =$
$(x + y)(x + y)(x + y) =$	$(x + y)(x + y)(x + y) =$

Problem 25 *Which algebra, Boolean or polynomial, does the following identity belong to? Why?*

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

Prove the identity.